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Generic Hopf–Neĭmark–Sacker bifurcations in feed-forward systems

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Abstract

We show that generic Hopf–Neĭmark–Sacker bifurcations occur in the dynamics of a large class of feed-forward coupled cell networks. To this end we present a framework for studying such bifurcations in parametrized families of perturbed forced oscillators near weak resonance points. Our approach is based on fine-tuning existing normal form techniques. We then apply this framework to show that certain cells in the feed-forward networks exhibit this forced oscillator dynamics and, hence, undergo generic Hopf–Neĭmark–Sacker bifurcations. These bifurcations correspond to the occurrence of resonance tongues in parameter space with a ‘standard’ geometry.

Mathematics Subject Classification: 37G15, 34C25, 34C27

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the general theory of dynamical systems an important problem has been, and to some extent still is, how one system of equations of motion can describe both very orderly (e.g. laminar) and extremely complicated (say, turbulent) behaviour. In the 1940s and 1950s Hopf, Landau and Lifschitz [25, 26, 28, 29] partially answered this question. A slightly different answer was proposed by Ruelle and Takens in the 1970s [30, 31]. Both approaches were reconciled and unified in [3] for further information also see [12, 16]. The idea is that the dynamical system at hand depends on parameters and that by repeated Hopf bifurcation the system gradually acquires ‘more frequencies’ and thus starts living on ever higher dimensional tori, which then occur as ‘centre manifolds’ in a possibly even much higher dimensional state space. The

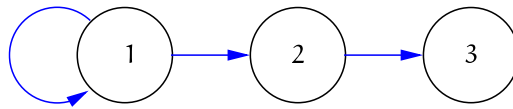


Figure 1. Three-cell linear feed-forward network.

dynamics on these tori can be either periodic, quasi-periodic or, in the case of flows from dimension three on, also chaotic.

One further question is whether this ‘generic’ scenario can also take place in certain subclasses of dynamical systems, such as coupled cell networks. This paper partly deals with this problem in the case of feed-forward systems. In particular we shall consider two stages of the scenario, concerning the Hopf bifurcation and the Hopf–Neïmark–Sacker bifurcation. The former of these deals with the bifurcation from stationary to periodic dynamics in the second cell, and the asymptotics of this Hopf bifurcation has turned out to be different from the general case [21]. The latter deals with the third cell and concerns the Hopf–Neïmark–Sacker bifurcation from a periodic orbit to an invariant two-torus. Here we show that the transition is ‘generic’ in the sense that the dynamics displays the characteristic alternation of quasi-periodic and phase-locked states, where the latter occurs in resonance tongues in parameter space.

Coupled cell systems. A coupled cell system is a network of dynamical systems, or ‘cells’, coupled together. These networks are represented by a directed graph with nodes representing cells and edges representing couplings between these cells. See, e.g. Golubitsky *et al* [23]. We consider the three-cell feed-forward network shown in figure 1, where the first cell is coupled externally to itself. The network represents the following system of ordinary differential equations (ODEs):

$$\begin{aligned}\dot{x}_1 &= F(x_1, x_1), \\ \dot{x}_2 &= F(x_2, x_1), \\ \dot{x}_3 &= F(x_3, x_2),\end{aligned}\tag{1}$$

with $x_j \in \mathbb{R}^2$. In our case, each cell satisfies the same dynamic law; only different choices of initial conditions may lead to different kinds of dynamics for each cell.

Synchrony breaking and quasiperiodicity in feed-forward networks. Golubitsky *et al* [21] describe synchrony-breaking Hopf bifurcations in a restricted class of feed-forward networks, with the first cell in equilibrium and the periods of the second and third cell being equal. Curiously, the amplitude of the periodic state in the second cell grows as $\lambda^{1/2}$, whereas the amplitude of the third cell grows as $\lambda^{1/6}$. Here λ is the ‘Hopf parameter’. Elmhirst and Golubitsky [19] show that this phenomenon occurs generically within this class of systems. The $\lambda^{1/6}$ growth rate is due to resonance; the third cell is forced periodically at a frequency near the Hopf frequency. McCullen *et al* [17] use this phenomenon of resonance near the Hopf bifurcation to demonstrate experimentally that the feed-forward network can be used as an efficient frequency filter/amplifier.

In [21] it is shown analytically that a generic Hopf bifurcation can occur from a periodic solution leading to a stable solution. Numerical examples show that the network dynamics may exhibit very different states in different cells. In a particular example, the dynamics corresponds to the first cell being in equilibrium, the second cell exhibiting time periodic behaviour, whereas the dynamics of the third cell is (conditionally) quasi-periodic.

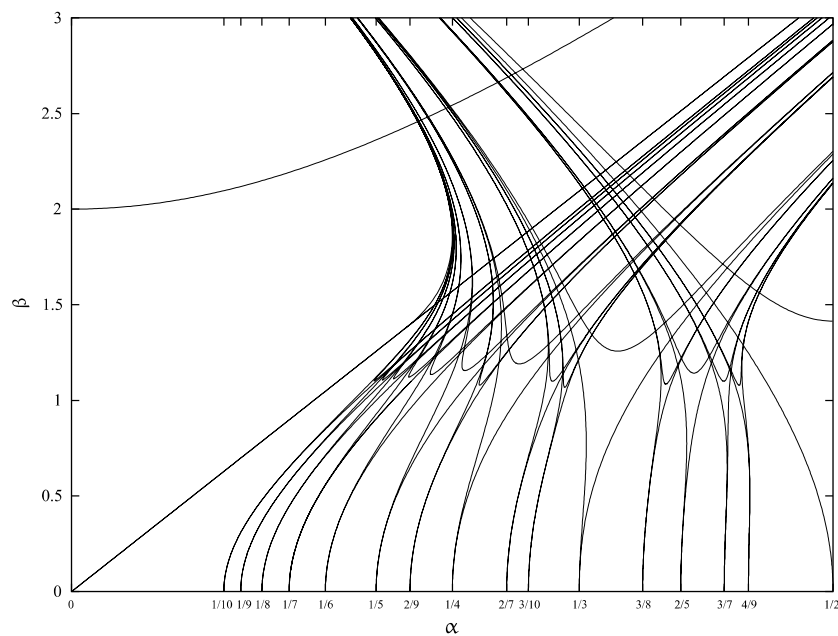


Figure 2. Resonance tongues in the Arnold-family [14]. The tongue boundaries correspond to saddle-node bifurcations. The interior of each tongue also contains a period-doubling bifurcation line.

Main results and contribution of the paper. It was generally believed that the dynamic phenomena in feed-forward networks are much more limited than those found in generic dynamical systems, but that, instead, the results of [21] describe generic behaviour of such networks. We refute this belief and show that such systems may exhibit fully generic Hopf–Neĭmark–Sacker bifurcations, with its characteristic alteration of quasi-periodic and phase-locked states [1, 5, 6, 12, 16, 27]. In particular, we show how this class of dynamic laws may give rise to time evolutions that are equilibria in cell 1, periodic in cell 2 and exhibit the generic Hopf–Neĭmark–Sacker phenomenon in cell 3. See also figures 2 and 4.

The detailed analysis of the dynamics of the third cell requires a fine-tuned version of a well-known normal form procedure. We show that a rather large class of families of perturbed forced oscillators, which contains the dynamics of our network cells as a special subclass, can be brought into a simple polynomial normal form near a $p : q$ resonance. As can be concluded from this normal form, Hopf–Neĭmark–Sacker bifurcations occur generically in such families.

Our analysis of cell dynamics requires detailed information on higher order terms of the normal form. Although the normal form procedure is well known and has been applied in various contexts [15, 27, 32, 33], the standard procedure does not provide this information. Therefore, in appendices A and B we extend the standard procedure considerably. Although not the primary topic of this paper, this part of our work can be applied in many settings involving forced oscillators and is, therefore, of independent interest.

Related work on resonance tongues. Heuristically speaking resonance occurs in the interaction of various oscillatory parts of a system, when the corresponding frequencies are rationally related. In our context the periodic dynamics of the second cell forces the dynamics of the third cell, where a ‘secondary’ Hopf bifurcation takes place, generating a second oscillatory

motion. We study the cases where these two motions are in certain resonances. As said earlier, we show that in the present coupled cell context a generic Hopf–Neĭmark–Sacker bifurcation occurs. It turns out that the subharmonic periodic motion, that with regard to its rotation number fits with the $p : q$ resonance at hand, occurs when the parameter ranges over a tongue-shaped region in parameter space. Along the boundaries of the tongue, the stable and unstable branches of the subharmonic motion annihilate each other in a saddle-node bifurcation. The tip of the tongue corresponds to a more degenerate bifurcation.

This kind of scenario occurs in many other situations also. An illustration of this is the Arnol'd family of circle maps [1], given by $x \mapsto x + 2\pi\alpha + \beta \sin x$. Here in the (α, β) plane tongues appear with their tips in $(\alpha, \beta) = (\frac{p}{q}, 0)$ and stretching out into the regions $\beta \neq 0$. See figure 2. The order of tangency of the tongue boundaries at the tip, as $q \rightarrow \infty$, has the same asymptotics as in the Hopf–Neĭmark–Sacker case. Also compare with [16, 18] and [5, 6].

Also in the worlds of Hamiltonian, reversible or equivariant systems resonance widely occurs. In all cases the resonant dynamics in parameter space is organized by the bifurcation set, which often contains tongues. As another example we refer to the parametrically forced pendulum, both in linear and nonlinear versions, where in the former case we deal with Hill's equation. For general reference, also including cases with quasi-periodic forcing, we refer to [2, 7, 9, 10, 11, 13, 15].

Overview of the paper. In section 2 we introduce the class of feed-forward networks of coupled cells, in which each cell has the same dynamics. For a large class of cell dynamics we analyse the time evolution of a linear chain of coupled cells, where each cell, except for the first one, is driven by the dynamics of its predecessor in the chain. We show that the third cell and its successors may exhibit forced oscillator dynamics, giving rise to generic Hopf–Neĭmark–Sacker bifurcations near weak resonance points. Section 3 presents a specific class of such linear networks in which such bifurcations occur. The resonance tongues in parameter space, a characteristic feature of such bifurcations, are analysed and turn out to exhibit the 'standard' geometry.

The technical details of the normal form techniques for the class of networks studied in this paper are presented in the appendices. Appendix A shows how to bring the linear part of the cell dynamics near the Hopf–Neĭmark–Sacker bifurcation point into the Jordan normal form. This step has to precede the application of the normal form procedure, which is presented in appendix B. In appendix C we present the geometric analysis of the resonance tongues in detail. Finally, appendix D shows that our class of networks satisfies the generic condition guaranteeing the occurrence of generic Hopf–Neĭmark–Sacker bifurcations.

2. Complicated dynamics in feed-forward networks

We present a large class of linear feed-forward networks of the form (1) consisting of identical cells, such that Hopf–Neĭmark–Sacker bifurcations occur generically in the third cell. This is inline with the behaviour observed numerically in a specific example presented in [21].

In our class of networks the first cell is in equilibrium and the second cell exhibits periodic behaviour. In section 2.2 we show that the third cell then has the dynamics of a forced oscillator. In section 2.3 we continue our analysis of this class of networks by showing that, under generic conditions, the third cell undergoes generic Hopf–Neĭmark–Sacker bifurcations. We do so by bringing the dynamics of this cell into the normal form, from which we easily derive the occurrence of resonance tongues. The proofs and details of these results are deferred to the appendices. In section 3 we establish the feasibility of our approach by presenting a large

class of examples that satisfy the generic conditions guaranteeing the occurrence of generic Hopf–Neĭmark–Sacker bifurcations.

2.1. Dynamics of the first two cells

\mathbb{S}^1 -equivariance. For simplicity we assume that the function F , describing the dynamics of each cell as in (1), is a perturbation of an \mathbb{S}^1 -equivariant function f in the sense that

$$F(z_2, z_1) = f(z_2, z_1) + \varepsilon P(z_2, z_1), \quad (2)$$

with

$$f(e^{i\theta} z_2, e^{i\theta} z_1) = e^{i\theta} f(z_2, z_1), \quad (3)$$

for all real θ . Here we identify the two-dimensional phase space of each cell with \mathbb{C} by writing $z_j = x_{j1} + ix_{j2}$. Furthermore, ε is a small perturbation parameter. The *unperturbed dynamics* of the network cells corresponds to the case $\varepsilon = 0$.

Identity (3) is a special assumption, also used in [21] to derive the curious phenomenon of decaying amplitudes of the periodic states in the second and third cells, for a certain class of dynamic laws. However, Elmhirst and Golubitsky [19] verify that this symmetry condition holds to third order after a change in coordinates. We have not been able to show that Hopf–Neĭmark–Sacker bifurcations occur in more general networks, i.e. in the case the unperturbed cell dynamics is not necessarily \mathbb{S}^1 -equivariant as in (3).

We also assume that the dynamics of each cell depends on external parameters $\mu \in \mathbb{R}^k$, to be specified later on. The \mathbb{S}^1 -equivariance condition (3) restricts the unperturbed cell dynamics to functions f of the form

$$f(z_2, z_1, \mu) = A(|z_1|^2, z_1 \bar{z}_2, |z_2|^2, \mu) z_1 + B(|z_1|^2, \bar{z}_1 z_2, |z_2|^2, \mu) z_2, \quad (4)$$

where A and B are complex functions depending on the parameter μ . We occasionally express the dependence of f , A and B on the parameter by writing f_μ , A_μ and B_μ , where $f_\mu(z_2, z_1) = f(z_2, z_1, \mu)$, etc.

Dynamics of the first and second cell. The \mathbb{S}^1 -equivariance condition (3) implies that $f_\mu(0, 0) = 0$. Assume that the linear part of $f_\mu(z_1, z_1)$ at $z_1 = 0$ has eigenvalues with a negative real part, i.e.

$$\operatorname{Re}(A_\mu(0) + B_\mu(0)) < 0, \quad (5)$$

for all parameter values μ in the region of our interest. Then the unperturbed dynamics $\dot{z}_1 = f_\mu(z_1, z_1)$ of the first cell has a stable equilibrium at $z_1 = 0$.

The second cell then has unperturbed dynamics

$$\dot{z}_2 = f_\mu(z_2, 0) = B(0, 0, |z_2|^2, \mu) z_2, \quad (6)$$

where we use that the first cell is in its stable equilibrium. Golubitsky and Stewart [22] introduce a large class of functions f_μ for which the second cell undergoes a Hopf bifurcation. For this class of cell dynamics, and for linear feed-forward networks of increasing length, there will be a ‘repeated Hopf’ bifurcation, reminiscent of the scenarios named after Landau–Lifschitz and Ruelle–Takens. In view of (6) the second cell undergoes a Hopf bifurcation at a parameter value $\mu_* \in \mathbb{R}^k$ if the following *additional condition* holds:

$$\begin{aligned} \operatorname{Re} B(0, 0, 0, \mu_*) &= 0, \\ D_\mu B(0, 0, 0, \mu_*) &\neq 0. \end{aligned} \quad (7)$$

In fact, (7) determines a hypersurface \mathcal{H}_* in parameter space \mathbb{R}^k , such that the second cell undergoes a Hopf bifurcation at every point of \mathcal{H}_* . In this case the periodic solutions emanating

from this bifurcation are circles of the form $|z_2| = r(\mu)$, where $r = r(\mu)$ is the non negative solution of the equation

$$\operatorname{Re} B(0, 0, r^2, \mu) = 0, \quad (8)$$

with $r(\mu_*) = 0$. Note that a (positive) real solution $r(\mu)$ exists (locally) only for parameter values in one of the closed half-spaces bounded by the hypersurface \mathcal{H}_* . Setting

$$\theta(\mu) = \operatorname{Im} B(0, 0, r(\mu)^2, \mu),$$

we see that the bifurcating periodic state of the second cell is

$$z_2(t) = r(\mu) e^{i\theta(\mu)t},$$

with $\theta(\mu_*) = \operatorname{Im} B(0, 0, 0, \mu_*)$. See also [21, section 5]. To guarantee that the equilibrium of the first cell and the periodic solution of the second cell persist for small values of the perturbation parameter ε we impose the condition

$$P(z_2, 0, \mu) \equiv 0, \quad (9)$$

where P is the perturbation term in (2). The periodic solution of the second cell will drive the dynamics of the third cell, which we consider next.

2.2. Forced oscillator dynamics of the third cell

The main topic of our research is the generic dynamics of the *third cell*, given the simple time evolutions of the first two cells introduced in the first part of this section. Here we like to know what are the correspondences and differences with the general ODE setting. In particular, this question regards the coexistence of periodic, quasi-periodic and chaotic dynamics.

In co-rotating coordinates the unperturbed dynamics of the third cell, i.e. the system $\dot{z}_3 = f_\mu(z_3, r(\mu)e^{i\theta(\mu)t})$, becomes time independent. To see this, set $z_3 = e^{i\theta(\mu)t} y$ and use the \mathbb{S}^1 -symmetry to derive

$$\begin{aligned} i\theta(\mu)e^{i\theta(\mu)t} y + e^{i\theta(\mu)t} \dot{y} &= f_\mu(e^{i\theta(\mu)t} y, r(\mu)e^{i\theta(\mu)t}) \\ &= e^{i\theta(\mu)t} f_\mu(y, r(\mu)). \end{aligned}$$

Therefore, the unperturbed dynamics of the third cell is given by

$$\dot{y} = -i\theta(\mu)y + f_\mu(y, r(\mu)). \quad (10)$$

Equation (10) is autonomous, so the present setting might exhibit Hopf bifurcations, as noted in [17], but this is still too simple to produce resonance tongues. Indeed, all (relative) periodic motion in (10) will lead to parallel (quasi-periodic) dynamics and the Hopf–Neĭmark–Sacker phenomenon. Therefore, we consider the perturbed family (2). In co-rotating coordinates $y = e^{-i\theta(\mu)t} z_3$ the perturbed dynamics of the third cell is given by

$$\dot{y} = -i\theta(\mu)y + f_\mu(y, r(\mu)) + \varepsilon e^{-i\theta(\mu)t} P(y e^{i\theta(\mu)t}, r(\mu) e^{i\theta(\mu)t}). \quad (11)$$

Scaling time by putting

$$\bar{t} = \theta(\mu)t \quad (12)$$

and denoting derivatives with respect to \bar{t} by primes, we see that the third cell therefore has *forced oscillator dynamics* with driving frequency 1:

$$y' = g(y, \mu) + \varepsilon G(y, \bar{t}, \mu), \quad (13)$$

where

$$g(y, \mu) = -iy + f(y, r(\mu), \mu) \quad (14)$$

and

$$G(y, \bar{t}, \mu) = e^{-i\bar{t}} P(y e^{i\bar{t}}, r(\mu) e^{i\bar{t}}).$$

In particular, G is 2π -periodic in \bar{t} . For simplicity of notation we omit the bars from now on, i.e. we again denote time by t .

The question about generic dynamics concerns the possible coexistence of periodic, quasi-periodic and chaotic dynamics, where the latter only is to be expected for larger values of the parameters (λ, ε) , cf [14].

We aim to investigate (11) for Hopf–Neĭmark–Sacker bifurcations, which are expected along codimension one strata \mathcal{H}_ε in the μ -space of parameters. In section 2.3 we explain the general context giving rise to the occurrence of such bifurcations. Subsequently, in section 3, we consider a specific example illustrating the procedure to verify the occurrence of such bifurcations.

2.3. Generic Hopf–Neĭmark–Sacker bifurcations in the third cell

Normal Form near the Hopf stratum. As said before, Hopf–Neĭmark–Sacker bifurcations in perturbed forced oscillators, such as (13), may occur subordinate to a non degenerate Hopf bifurcation in the *unperturbed* system. So consider $(y_0, \mu_0) \in \mathbb{C} \times \mathbb{R}^k$ such that $g(y_0, \mu_0) = 0$ and $\text{tr}(d_y g(y_0, \mu_0)) = 0$, with $\det(d_y g(y_0, \mu_0)) > 0$. According to the implicit function theorem, the equation $g(y, \mu) = 0$ has a unique solution

$$y = y_\mu, \tag{15}$$

for μ near μ_0 , coinciding with y_0 for $\mu = \mu_0$. Let $a(\mu) \pm i\omega(\mu)$ be the eigenvalues of the linear part of $\dot{y} = g(y, \mu)$ at the equilibrium point y_μ , for μ near μ_0 . Then

$$a(\mu) = \frac{1}{2} \text{tr}(d_y g(y_\mu, \mu)) \quad \text{and} \quad \omega(\mu) = \sqrt{\det(d_y g(y_\mu, \mu)) - a(\mu)^2}. \tag{16}$$

The system $y' = g(y, \mu)$, corresponding to the unperturbed dynamics of the third cell, undergoes a non degenerate Hopf bifurcation near μ_0 , if the following (versality) condition holds:

Versal unfolding of linear part condition:

$$\text{The map } \mathbb{R}^k \rightarrow \mathbb{R}^2 : \mu \mapsto (a(\mu), \omega(\mu)) \text{ has rank two at } \mu_0. \tag{17}$$

If this condition holds, the non degenerate Hopf bifurcation occurs when the parameter μ passes a codimension one hypersurface \mathcal{H}_0 consisting of parameters μ at which $a(\mu) = 0$. Note that the zero set of a is a regular hypersurface in parameter space near μ_0 . When μ crosses \mathcal{H}_0 a periodic orbit of period $\omega(\mu)$ bifurcates (i.e. is born or dies) from the equilibrium point y_μ .

Subharmonics may emerge or disappear if the parameters are close to a *resonance point*, a point on the Hopf stratum at which the normal frequency is rational. Since the versal unfolding of linear part condition guarantees that the normal frequency $\omega(\mu)$ is non-constant as μ ranges over the Hopf stratum \mathcal{H}_0 , there is an abundance of such points on \mathcal{H}_0 . Select $\mu_0 \in \mathcal{H}_0$ such that $\omega(\mu_0) = \frac{p}{q}$, where p and q are relatively prime. In other words, the system exhibits a $p : q$ resonance for $\mu = \mu_0$. We only consider *weak resonances*, i.e. we assume that $q \geq 5$. See [1] and [33] for a discussion of strong and weak resonances.

We are now ready to state one of our main results on the normal form for the forced oscillator dynamics of the third cell. This normal form will reveal the generic Hopf–Neĭmark–Sacker bifurcations in the dynamics of this cell.

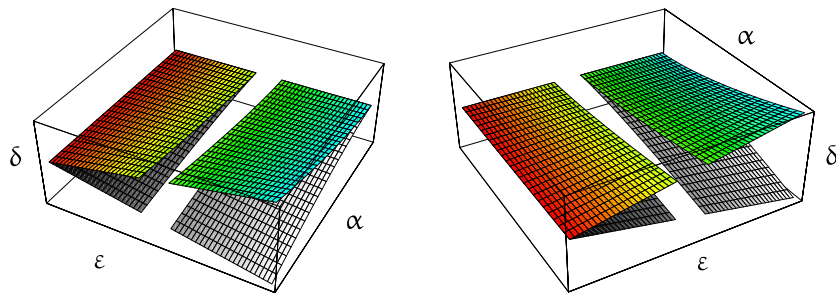


Figure 3. Resonance zones for forced oscillator families: the Hopf–Neimark–Sacker phenomenon (sliced and presented from two different viewpoints for enhanced visualization).

Theorem 1 (normal Form to order q). Suppose the forced oscillator (13) satisfies the versal unfolding of linear part condition, for μ near μ_0 , and the normal frequency at μ_0 is equal to $\omega_0 = \frac{p}{q}$. Then the system can be brought into the normal form

$$Z' = i\omega_0 Z + (\alpha + i\delta) Z + Z F(|Z|^2, v, \varepsilon) + \gamma(v, \varepsilon) \varepsilon \overline{Z}^{q-1} e^{ipt} + O(|Z|^q), \quad (18)$$

locally uniformly in (v, ε) . Here $F(|Z|^2, v, \varepsilon)$ is a complex polynomial of degree at most $q - 1$ with $F(0, v, \varepsilon) \equiv 0$. The $O(|Z|^q)$ terms are 2π -periodic in t . Furthermore, $Z \in \mathbb{C}$, and $v \in \mathbb{R}^k$ is a new parameter obtained by a reparametrization $\mathbb{R}^k \rightarrow \mathbb{R}^k$, with $v_1 = \alpha$ and $v_2 = \delta$. Let $v_0 \in \mathbb{R}^k$ correspond to the parameter value μ_0 under this reparametrization. Generically, $\gamma(v_0, 0) \neq 0$, and in this case a further reparametrization brings the system into the form

$$Z' = i\omega_0 Z + (\alpha + i\delta) Z + Z F(|Z|^2, v, \varepsilon) + \varepsilon \overline{Z}^{q-1} e^{ipt} + O(|Z|^q). \quad (19)$$

The normal form (19) is \mathbb{Z}_q -symmetric and even S^1 -symmetric in the unperturbed case corresponding to $\varepsilon = 0$. Note that the linear part is a versal unfolding of the linear part $Z' = i\omega_0 Z$ of the central system corresponding to $(v, \varepsilon) = (v_0, 0)$.

The proof of theorem 1 is deferred to appendix B. Theorem 3 gives a more precise expression for the constant $\gamma(v_0, 0)$, showing that, generically, this constant is indeed non zero. Here we note that the normal form transformation also involves a change in parameters. The hyperplane $\varepsilon = 0$ in parameter space is invariant under this change of parameters, so ε is again a small perturbation parameter for the normalized system.

Resonance tongues. The geometry of the resonance tongues can be deduced from the normal form (19) of the forced oscillator dynamics of the third cell. The following theorem implies that, under generic additional conditions, the order of tangency at the tongue tips is generic. Here the Hopf coefficient $F_u(0, 0, 0)$ is the partial derivative of $F(u, \mu, \varepsilon)$ with respect to u . It is the coefficient of $z^2 \bar{z}$ in (18) for $(v, \varepsilon) = (v_0, 0)$. Note that, for $\varepsilon = 0$, the system undergoes a generic Hopf bifurcation if the real part of this coefficient is non zero, cf [24, chapter 3.4].

Theorem 2 (geometry of the resonance tongues). Consider system (19). If $F_u(0, 0, 0) \neq 0$ and $\gamma(v_0, 0) \neq 0$ in (18), then the $p : q$ resonance tongues of the forced oscillator (19) are subsets of parameter space bounded by hypersurfaces of the form

$$\delta = \pm \varepsilon (-\alpha)^{(q-2)/2} + O(\varepsilon^2). \quad (20)$$

The boundary of the resonance tongues is shown in figure 3, for a three-parameter family, in the case $q = 5$. In fact, the essential parameters of the forced oscillator (18) are α, δ

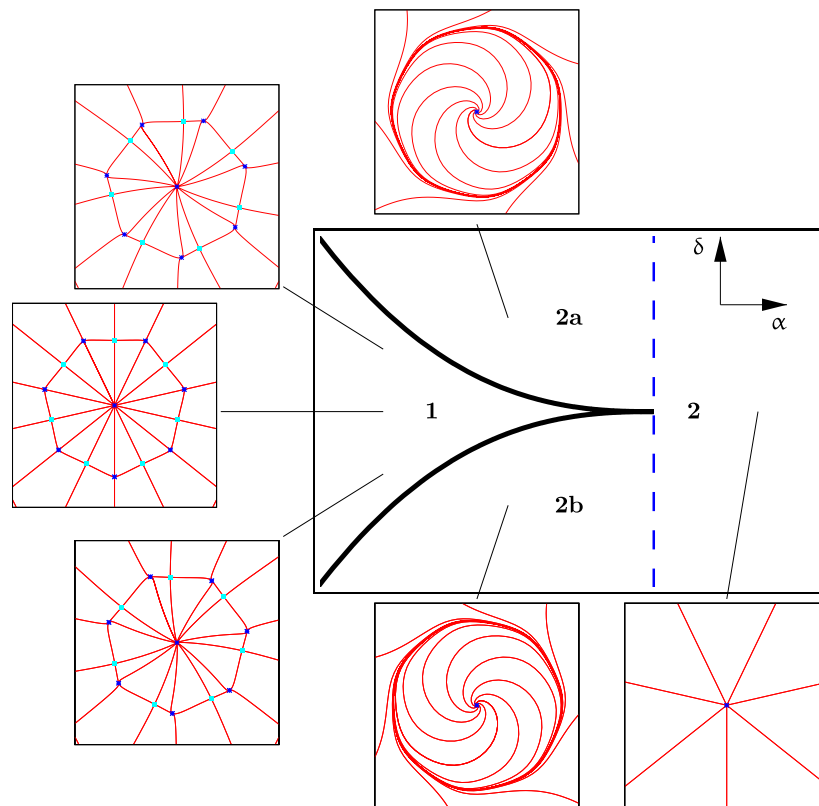


Figure 4. Takens normal form vector field of the Poincaré map of the periodic system $\dot{z} = i\omega_0 z + (\alpha + i\delta)z + z|z|^2 + \varepsilon \bar{z}^4 e^{i\tau}$, corresponding to system (19) in the weakly resonant case $p = 1$ and $q = 7$, and fixed $\varepsilon \neq 0$. The Poincaré map itself is obtained by composing the time-1-flow of this vector field by a rotation over $2\pi/7$. See appendix C and [33]. The normally hyperbolic invariant circle of the Poincaré map corresponds to an invariant torus in the 3D phase space $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$. The dynamics on the invariant circle undergoes a saddle-node bifurcation when the parameters pass the boundary of the resonance tongue, corresponding to the birth (or death) of a pair of periodic orbits of period q .

and ε . Together, these tongue boundaries form a cuspidal surface. At this surface the Hopf–Neïmark–Sacker bifurcation should occur, since here the Floquet exponents of the linear part of the forced oscillator cross the complex unit circle. This bifurcation gives rise to an invariant 2-torus in the 3D phase space $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$; see figure 4. For a more detailed bifurcation analysis of generic and mildly degenerate Hopf–Neïmark–Sacker families we refer to our recent work [8]. More specifically, resonances occur when the eigenvalues of the Poincaré time- 2π -map cross the unit circle at roots of unity $e^{2\pi i p/q}$. The normal form of the Poincaré map and, hence, the bifurcation diagram shown in figure 4 are similar to those obtained by Gambaudo in [20, equation (3.1) and figure 2]. Also, the ‘resonance horn’ of [20, figure 1] is similar to the resonance set shown in figure 3. ‘Inside’ the tongue the 2-torus is phase-locked to subharmonic periodic solutions of order q [14, 16].

Remark. In a feed-forward network with more cells, in the higher order cells we expect the generic and standard Hopf–Landau–Lifschitz–Ruelle–Takens scenarios with respect to the coexistence of periodicity, quasi-periodicity and chaos [12, 16, 30, 31].

Genericity of the normal form. The generic conditions guaranteeing that the constant $\gamma(v_0, 0)$ in (18) be non zero, and hence that the normal form of the dynamics of the third cell is of the form (19), can be made explicit in specific examples of cell dynamics. These conditions are expressed in terms of the coefficients of a low-degree jet of f and the coefficients of certain resonances in the perturbation term P of the forced oscillator dynamics (2) of the third cell. In section 3 we present a class of examples of cell dynamics of the form (13), where the unperturbed system is a low-degree polynomial system, and show what these generic conditions reduce to for this class of systems.

To derive the conditions guaranteeing $\gamma(v_0, 0) \neq 0$, cf theorem 1, we first introduce some notation. The Jordan normal form of the linear part of $\dot{y} = g(y, \mu_0)$ at $y = y_0$ is of the form $\dot{z} = (a(\mu) + i\omega(\mu))z$, cf (16). This linear part, denoted by L_μ , is of the form

$$L_\mu(w) = \alpha(\mu) w + \beta(\mu) \bar{w}.$$

Note that

$$a(\mu) = \operatorname{Re} \alpha(\mu).$$

Writing $\alpha(\mu) = a(\mu) + ib(\mu)$ and $\beta(\mu) = c(\mu) + id(\mu)$, we observe that

$$\omega(\mu) = \sqrt{\det(L_\mu) - a(\mu)^2} = \sqrt{b(\mu)^2 - c(\mu)^2 - d(\mu)^2}.$$

A straightforward computation shows that the transformation s_μ , bringing L_μ into the Jordan normal form, i.e. such that

$$s_\mu^{-1} \circ L_\mu \circ s_\mu(Z) = (i\omega(\mu) + a(\mu))Z,$$

is given by

$$s_\mu(Z) = \sigma(\mu)Z + \tau(\mu)\bar{Z}, \quad (21)$$

where

$$\begin{aligned} \sigma(\mu) &= \frac{1}{2}(b(\mu) - d(\mu) + \omega(\mu)) + \frac{1}{2}ic, \\ \tau(\mu) &= \frac{1}{2}(b(\mu) - d(\mu) - \omega(\mu)) + \frac{1}{2}ic. \end{aligned} \quad (22)$$

We require that the linear transformation s_μ is invertible, i.e. $\Delta(\mu) := \operatorname{Det} s_\mu = |\sigma(\mu)|^2 - |\tau(\mu)|^2 \neq 0$. Its inverse is then given by

$$s_\mu^{-1}(Y) = \Delta(\mu)^{-1}(\overline{\sigma(\mu)}Y - \tau(\mu)\bar{Y}).$$

To simplify notation we put $r_0 = r(\mu_0)$, with r given by (8), $y_0 = y(\mu_0)$, $\sigma_0 = \sigma(\mu_0)$, $\tau_0 = \tau(\mu_0)$ and $\Delta_0 = \Delta(\mu_0)$.

Theorem 3 (coefficient of rotationally non-symmetric normal form term). *Suppose the conditions of theorem 1 hold. Then the coefficient $\gamma(v_0, 0)$ of the non-symmetric term $\varepsilon \bar{z}^{q-1} e^{ipt}$ in the normal form (18) is non zero for generic f and P , cf (2). Furthermore, this coefficient is of the form*

$$\gamma(v_0, 0) = c_0(f) + c_1(f) C_1(P) + c_2(f) C_2(P) + C(P), \quad (23)$$

where

- $c_0(f)$ depends on the $(q-1)$ -jet of $y \mapsto f(y, r_0, \mu_0)$ at $y = y_0$;
- $c_1(f)$ and $c_2(f)$ depend linearly on f , and $c_1(f) = c_2(f) = 0$ if f is a polynomial of degree less than q in z, \bar{z} ;

- $C_1(P)$ and $C_2(P)$ are linear in P ;
- $C(P)$ is given by

$$C(P) = \frac{1}{2\pi (q-1)!} \int_0^{2\pi} \frac{\partial^{q-1}}{\partial z^{q-1}} \Big|_{z=0} e^{-ipt} \left(\overline{\sigma_0} \Lambda_P(z, t) - \tau_0 \overline{\Lambda_P(z, t)} \right) dt, \quad (24)$$

where

$$\Lambda_P(z, t) = \Delta_0^{-1} e^{-it} P((y_0 + \sigma_0 z + \tau_0 \bar{z}) e^{it}, r_0 e^{it}). \quad (25)$$

In other words, the contribution $C(P)$ of the perturbation term P is the coefficient of the term $\bar{z}^{q-1} e^{ipt}$ in the Taylor–Fourier expansion of the function $(z, t) \mapsto \overline{\sigma_0} \Lambda_P(z, t) - \tau_0 \overline{\Lambda_P(z, t)}$.

In general, it is hard to determine the constant $c_0(f)$ explicitly, although in principle the normal form procedure described in appendix B.1 gives an explicit method for doing so. On the other hand, the constant $C(P)$ can often be computed more easily, as we will show in the case study in section 3. Furthermore, for generic perturbation terms P we have $C(P) \neq 0$. If f is a polynomial of degree less than q in z, \bar{z} , we have $\gamma(v_0, 0) = c_0(f) + C(P) \neq 0$ for generic P . We will analyse this situation in more detail in our case study.

A proof of theorem 3 is given in appendix D.

3. Generic Hopf–Neĭmark–Sacker bifurcations in feed-forward networks: a case study

We present a class of explicit examples that satisfy the generic conditions guaranteeing the occurrence of generic Hopf–Neĭmark–Sacker bifurcations.

Cell dynamics in the unperturbed setting. Condition (3) of \mathbb{S}^1 -symmetry, i.e.

$$f(e^{i\theta} z_2, e^{i\theta} z_1) = e^{i\theta} f(z_2, z_1),$$

restricts the unperturbed cell dynamics to functions f of the form (4), i.e.

$$f(z_2, z_1, \mu) = A(|z_1|^2, z_1 \bar{z}_2, |z_2|^2, \mu) z_1 + B(|z_1|^2, \bar{z}_1 z_2, |z_2|^2, \mu) z_2.$$

In our case study we simply take

$$A = -1 \quad \text{and} \quad B = \lambda + i + (1 + \kappa i)|z_1|^2 - |z_2|^2.$$

Here κ is a second parameter, so from now on $\mu = (\lambda, \kappa) \in \mathbb{R}^2$. With these choices we have

$$f_\mu(z_2, z_1) = -z_1 + (\lambda + i + (1 + \kappa i)|z_1|^2 - |z_2|^2) z_2. \quad (26)$$

With this choice of f condition (5) is satisfied, so the first cell has a stable equilibrium at $z_1 = 0$. The dynamics of the second cell is

$$\dot{z}_2 = f_\mu(z_2, 0) = (\lambda + i - |z_2|^2) z_2,$$

so the second cell undergoes a supercritical Hopf bifurcation at $\lambda = 0$, cf (8), with frequency $\theta(\mu) \equiv 1$. In other words, in this case study we do not need to rescale time as in (12). The hypersurface \mathcal{H}_* , defined by (7), is of the form $\mathcal{H}_* = \{(0, \kappa) \mid \kappa \in \mathbb{R}\}$. The stable periodic solution, occurring for $\lambda > 0$, has the form

$$z_2(t) = \sqrt{\lambda} e^{it},$$

in other words, $r(\mu) = \sqrt{\lambda}$, in terms of (8). In particular, we restrict the parameters to the half space $\lambda > 0$ from now on. In view of (13) the dynamics of the third cell is of the form

$$\dot{y} = g(y, \mu) + \varepsilon e^{-it} P(y e^{it}, \sqrt{\lambda} e^{it}), \quad (27)$$

with

$$g(y, \mu) = -iy + f_\mu(y, \sqrt{\lambda}) = -\sqrt{\lambda} + (2 + i\kappa)\lambda y - |y|^2 y. \quad (28)$$

We first determine the curve in the (λ, κ) -plane corresponding to the occurrence of Hopf bifurcations of the unperturbed dynamics of the third cell (i.e. with $\varepsilon = 0$). Subsequently, we present several examples of perturbed dynamics of this cell for which generic Hopf–Neimark–Sacker bifurcations occur.

Generic Hopf bifurcation of the unperturbed system. Before specifying the perturbation term in (11), we determine the Hopf stratum for the unperturbed dynamics.

Lemma 4. *The Hopf stratum of the unperturbed dynamics of the third cell, corresponding to the family (27) with $\varepsilon = 0$, is the curve \mathcal{H}_0 in the (λ, κ) plane given by*

$$\lambda = \frac{1}{\sqrt{\kappa^2 + 1}}, \quad \text{with} \quad |\kappa| > 1.$$

If $\mu = (\lambda, \kappa) \in \mathcal{H}_0$, the Hopf bifurcation takes place at the point

$$y(\mu) = \frac{1 - \kappa i}{(1 + \kappa^2)^{\frac{3}{4}}},$$

and the normal frequency of the Hopf bifurcation along this Hopf curve is given by

$$\omega(\kappa) = \sqrt{\frac{\kappa^2 - 1}{\kappa^2 + 1}}.$$

Proof. The Hopf stratum is the curve given by the equations

$$g(y, \mu) = 0 \quad \text{and} \quad \text{tr}(d_y g(y, \mu)) = 0,$$

with $\det(d_y g(y, \mu)) > 0$. Since $d_y g(y, \mu)w = \alpha w + \beta \bar{w}$, with

$$\alpha = (2 + i\kappa)\lambda - 2|y|^2 \quad \text{and} \quad \beta = -y^2, \quad (29)$$

and $\text{tr}(d_y g(y, \mu)) = 2\text{Re } \alpha$, these equations reduce to

$$\begin{cases} -\sqrt{\lambda} + (2 + i\kappa)\lambda y - |y|^2 y = 0, \\ \lambda - |y|^2 = 0. \end{cases}$$

This set of equations gives the expressions for λ and y in terms of κ . We also have to guarantee that $\det(d_y g(y, \mu)) > 0$ at a point $(y, \mu) \in \mathcal{H}_0$. A short calculation shows that

$$\det(dg_\mu) = |2(\lambda - |y|^2)^2 + i\kappa\lambda|^2 - |y|^4 = \lambda\kappa^2\lambda^2 - |y|^4 = \frac{\kappa^2 - 1}{\kappa^2 + 1},$$

so the frequency along the Hopf curve \mathcal{H}_0 is given by the expression for $\omega(\kappa)$ stated in the lemma, subject to the condition $|\kappa| > 1$. \square

This result shows that the normal frequency varies monotonically from 0 to 1 along the Hopf curve \mathcal{H}_0 . We study Hopf–Neimark–Sacker bifurcations near a Hopf point μ_0 at which $\omega(\mu_0) = \frac{p}{q}$, where p and q are relatively prime and $0 < p < q$. According to lemma 4, this happens at the points $\mu_0 = (\lambda_0, \kappa_0)$, with

$$\kappa_0 = \pm \sqrt{\frac{1 - \omega(\mu_0)^2}{1 + \omega(\mu_0)^2}} = \pm \sqrt{\frac{q^2 - p^2}{q^2 + p^2}}, \quad \lambda_0 = \sqrt{\frac{1}{2} \left(1 - \frac{p^2}{q^2}\right)}.$$

If the system satisfies the versal unfolding of linear part condition, theorem 1 yields the normal form (18).

Lemma 5. *The system $y' = g(y, \mu)$, with g given by (28), satisfies the versal unfolding of linear part condition, i.e. the map*

$$(\lambda, \kappa) \mapsto (a(\lambda, \kappa), \omega(\lambda, \kappa))$$

with a and ω given by (16) is invertible, locally near any point of the Hopf stratum \mathcal{H}_0 .

Proof. Let (λ_0, κ_0) be a point on the Hopf stratum \mathcal{H}_0 . We have to prove that

$$\left. \frac{\partial(a, \delta)}{\partial(\lambda, \kappa)} \right|_{(\lambda_0, \kappa_0)} \neq 0.$$

Putting $Y = |y_{\lambda, \kappa}|^2$ we get $a = 2(\lambda - Y)$ and $\delta = \lambda^2 \kappa^2 - Y^2 - \omega_N^2$. Therefore

$$\begin{aligned} \frac{\partial(a, \delta)}{\partial(\lambda, \kappa)} &= \begin{vmatrix} 2 \left(1 - \frac{\partial Y}{\partial \lambda} \right) & -2 \frac{\partial Y}{\partial \kappa} \\ 2 \left(\lambda \kappa^2 - Y \frac{\partial Y}{\partial \lambda} \right) & 2 \left(\lambda^2 \kappa - Y \frac{\partial Y}{\partial \kappa} \right) \end{vmatrix} \\ &= 4 \left(\lambda^2 \kappa - \lambda^2 \kappa \frac{\partial Y}{\partial \lambda} + \lambda \kappa^2 \frac{\partial Y}{\partial \kappa} - Y \frac{\partial Y}{\partial \kappa} \right). \end{aligned}$$

So it remains to compute the values of $\frac{\partial Y}{\partial \lambda}$ and $\frac{\partial Y}{\partial \kappa}$ at (λ_0, κ_0) . Now we pass to real coordinates by putting $y = u + iv$, with $u, v \in \mathbb{R}$. Denote the real and imaginary parts of $g(u + iv)$ by $g_r(u, v)$ and $g_i(u, v)$, respectively. Since $g(y_{\lambda, \kappa}) = 0$, we have $Y = u^2 + v^2$, where u, v is the solution of the system of equations

$$g_r(u, v) = g_i(u, v) = 0. \quad (30)$$

Note that u and v depend on λ and κ and that

$$\frac{\partial Y}{\partial \lambda} = 2 \left(u \frac{\partial u}{\partial \lambda} + v \frac{\partial v}{\partial \lambda} \right) \quad \text{and} \quad \frac{\partial Y}{\partial \kappa} = 2 \left(u \frac{\partial u}{\partial \kappa} + v \frac{\partial v}{\partial \kappa} \right). \quad (31)$$

Differentiating each of the equations in (30) with respect to λ and κ we obtain

$$\begin{pmatrix} \frac{\partial g_r}{\partial u} & \frac{\partial g_r}{\partial v} \\ \frac{\partial g_i}{\partial u} & \frac{\partial g_i}{\partial v} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial \lambda} & \frac{\partial u}{\partial \kappa} \\ \frac{\partial v}{\partial \lambda} & \frac{\partial v}{\partial \kappa} \end{pmatrix} + \begin{pmatrix} \frac{\partial g_r}{\partial \lambda} & \frac{\partial g_r}{\partial \kappa} \\ \frac{\partial g_i}{\partial \lambda} & \frac{\partial g_i}{\partial \kappa} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

From this identity we compute the values of $\frac{\partial u}{\partial \lambda}$, $\frac{\partial u}{\partial \kappa}$, $\frac{\partial v}{\partial \lambda}$ and $\frac{\partial v}{\partial \kappa}$ at (λ_0, κ_0) , using *Mathematica*. Plugging these values into (31), and using

$$\lambda_0 = (1 + \kappa_0^2)^{-\frac{1}{2}}, \quad u_0 = (1 + \kappa_0^2)^{-\frac{3}{4}} \quad \text{and} \quad v_0 = -\kappa_0 (1 + \kappa_0^2)^{-\frac{3}{4}},$$

we obtain

$$\left. \frac{\partial(a, \delta)}{\partial(\lambda, \kappa)} \right|_{(\lambda_0, \kappa_0)} = \frac{16\kappa_0}{\kappa_0^4 - 1} > 0.$$

From this inequality we conclude that the reparametrization is locally invertible at points (λ_0, κ_0) of the Hopf curve. \square

The latter lemma allows us to conclude that the dynamics of the third cell can be brought into the normal form (18). Moreover, it turns out that we can explicitly compute the coefficient $\gamma(v_0, 0)$ in this normal form. Hence, theorem 1 guarantees that the normal form can be reduced even further to the simpler form (19).

The Hopf coefficient. In our case study, the second condition for the occurrence of generic Hopf–Neĭmark–Sacker bifurcations also holds, i.e. the real part of the Hopf coefficient is non zero.

Lemma 6. *In the case study (27), the real part of the Hopf coefficient $F_u(0, 0, 0)$, or, equivalently, the real part of the coefficient of $z^2\bar{z}$ in the normal form (18) for $(\mu, \varepsilon) = (\mu_0, 0)$, is non zero.*

Proof. We first bring the linear part of the system $\dot{y} = g(y, \mu_0)$ into the Jordan normal form, with g given by (28) with $\lambda = (\kappa^2 + 1)^{-1/2}$ and $|\kappa| > 1$, cf lemma 4. To this end, first bring the Hopf singular point to the origin by the translation $Y = y - y_0$, where $y_0 = y(\mu_0)$. This translation transforms the system $\dot{y} = g(y, \mu_0)$ into the form

$$\dot{Y} = B_{10}Y + B_{01}\bar{Y} + B_{20}Y^2 + B_{11}Y\bar{Y} + B_{02}\bar{Y}^2 - Y^2\bar{Y}, \quad (32)$$

with

$$\begin{aligned} B_{10} &= \lambda_0(2 + i\kappa_0) - 2|y_0|^2, & B_{01} &= -y_0^2, \\ B_{20} &= -\bar{y}_0, & B_{11} &= -2y_0, & B_{02} &= 0. \end{aligned}$$

Subsequently, we bring the linear part of (32) into the Jordan normal form $\dot{Z} = i\omega_0 Z$ via (21), i.e. we put

$$Y = \sigma_0 Z + \tau_0 \bar{Z},$$

where σ_0 and τ_0 are given by (22) for $\mu = \mu_0$. In view of (29) we have

$$\begin{aligned} a_0 &= \operatorname{Re} \alpha(\mu_0) = 0, & b_0 &= \operatorname{Im} \alpha(\mu_0) = \kappa_0 \lambda_0, \\ c_0 &= \operatorname{Re} \beta(\mu_0) = -\operatorname{Re} y_0^2, & d_0 &= \operatorname{Im} \beta(\mu_0) = -\operatorname{Im} y_0^2, \end{aligned} \quad (33)$$

where $y_0 = (1 - \kappa_0 i)(\kappa_0^2 + 1)^{-3/4}$ and $\lambda_0 = (1 + \kappa_0^2)^{-1/2}$, cf lemma 4. A straightforward computer-assisted calculation shows that the transformed system is of the form

$$\dot{Z} = i\omega_0 Z + A_{20}Z^2 + A_{11}Z\bar{Z} + A_{02}\bar{Z}^2 + A_{30}Z^3 + A_{21}Z^2\bar{Z} + A_{12}Z\bar{Z}^2 + A_{03}\bar{Z}^3, \quad (34)$$

where

$$\begin{aligned} A_{20} &= \frac{\kappa_0}{2(1 + \kappa_0^2)^{5/4}}(1 - 2i\sqrt{\kappa_0^2 - 1}), \\ A_{11} &= \frac{\kappa_0}{(1 + \kappa_0^2)^{5/4}}(-1 + 2i\sqrt{\kappa_0^2 - 1}), \\ A_{02} &= \frac{\kappa_0}{2(1 + \kappa_0^2)^{5/4}}(2 - \kappa_0^2 - 2i\sqrt{\kappa_0^2 - 1}), \\ A_{21} &= \frac{\kappa_0^2 - \kappa_0^4}{(1 + \kappa_0^2)^2}. \end{aligned} \quad (35)$$

The values of A_{30} , A_{12} and A_{03} are irrelevant, in view of the following result: the system (34) is brought into the normal form (18), which for $(\kappa, \lambda, \varepsilon) = (\kappa_0, (\kappa_0^2 + 1)^{-1/2}, 0)$ is given by

$$\dot{z} = i\omega_0 z + c_1 z^2 \bar{z} + O(|z|^4),$$

where the *Hopf coefficient* c_1 is given by

$$c_1 = \frac{i}{2\omega_0}(2A_{20}A_{11} - 2|A_{11}|^2 - \frac{4}{3}|A_{02}|^2) + A_{21}. \quad (36)$$

For a derivation of this result, see [27, lemma 3.6], especially (3.18) in the proof of this lemma, or [24, appendix to section 3.4], especially (3.4.11) and (3.4.26). In fact, the normal

form procedure presented in appendix B.1 yields the same expression for the Hopf coefficient. Substituting the values of A_{ij} given in (35) into (36) yields

$$c_1 = -\frac{\kappa_0^2}{1 + \kappa_0^2} + \frac{\kappa_0^2 - 6\kappa_0^4}{3(\kappa_0^2 + 1)^2 \sqrt{\kappa_0^2 - 1}} i,$$

and hence $\operatorname{Re} c_1 \neq 0$. \square

The perturbation term. We continue our case study with a closer analysis of the perturbation term in (27). We show that, for a general class of perturbation terms P in (2), the constant $C(P)$, introduced in theorem 3, can be chosen arbitrarily. Since the coefficient $\gamma(v_0, 0)$ of the rotationally non-symmetric term in the normal form (18) is equal to $\gamma(v_0, 0) = C_f + C(P)$, this implies that a suitable choice of the perturbation term P yields $\gamma(v_0, 0) \neq 0$. In view of theorem 2, this implies the occurrence of generic Hopf–Neĭmark–Sacker bifurcations in the dynamics of the third cell.

Consider a perturbation term of the monomial form

$$P(z_2, z_1) = P_0 z_1^k \bar{z}_1^l z_2^m \bar{z}_2^{q-1-m},$$

where P_0 is a non zero complex constant. Note that this term satisfies condition (9), guaranteeing the persistence of the equilibrium of the first cell and the periodic solution of the second cell.

Then, in the notation of theorem 3,

$$\begin{aligned} \Delta_P(z, t) &= P_0 \lambda_0^{\frac{1}{2}(k+l)} e^{i(k-l+2m-q)t} (y_0 + \sigma_0 z + \tau_0 \bar{z})^m (\bar{y}_0 + \bar{\sigma}_0 \bar{z} + \bar{\tau}_0 \bar{z})^{q-1-m} \\ &= P_0 \lambda_0^{\frac{1}{2}(k+l)} e^{i(k-l+2m-q)t} (\tau_0^m \bar{\sigma}_0^{q-1-m} \bar{z}^{q-1} + \sigma_0^m \bar{\tau}_0^{q-1-m} z^{q-1}) \\ &\quad + z \bar{z} R_{q-3}(z, t) + R_{q-2}(z, t), \end{aligned}$$

where $R_k(z, t)$ is a polynomial of degree k in z and \bar{z} , with coefficients that are 2π -periodic in t . Since the term $z \bar{z} R_{q-3}(z, t) + R_{q-2}(z, t)$ does not contain a term of the form $\bar{z}^{q-1} e^{ip t}$, it does not contribute to the coefficient $C(P)$, cf theorem 3. More precisely,

$$C(P) = \begin{cases} P_0 \lambda_0^{\frac{1}{2}(k+l)} \bar{\sigma}_0^{q-m} \tau_0^m, & \text{if } k - l + 2m - q = p, \\ -\bar{P}_0 \lambda_0^{\frac{1}{2}(k+l)} \bar{\sigma}_0^m \tau_0^{q-m}, & \text{if } k - l + 2m - q = -p, \\ 0, & \text{if } k - l + 2m - q \neq \pm p. \end{cases}$$

Given the value of C_f , we take $k - l + 2m - q = \pm p$ and choose P_0 such that $\gamma(v_0, 0) = C_f + C(P) \neq 0$. This choice of P_0 is possible since λ_0 , σ_0 and τ_0 are non zero. Indeed, $\lambda_0 = (\kappa_0^2 + 1)^{-1/2} \neq 0$ and $\operatorname{Im} \sigma_0 = \operatorname{Im} \tau_0 = c_0 \neq 0$, cf (22) and (33).

Remark. Observe that $C(P) = 0$ if the degree of P in z_2, \bar{z}_2 is less than $q - 1$. Therefore, our example above is the ‘simplest’ yielding non zero $C(P)$.

Consequently, for this choice of P the dynamics exhibits Hopf–Neĭmark–Sacker bifurcations corresponding to the birth and death of subharmonics of order q , for parameter values $(\lambda, \kappa, \varepsilon)$ near the point $(\lambda_0, \kappa_0, 0)$ on the Hopf line in the plane $\varepsilon = 0$ in parameter space. This phenomenon even occurs for generic perturbation terms P .

Lemma 7. For generic choices of the perturbation term P in (27) the coefficient $\gamma(v_0, 0)$ of the rotationally non-symmetric term in the normal form (18) for $(v, \varepsilon) = (v_0, 0)$ is non zero.

Proof. Since $C(P)$ is linear in P , we have $C(P) \neq 0$ for generic P . To see this, decompose P as $P = P' + P'' + P'''$, with P' of degree less than $q - 1$, P'' homogeneous of degree $q - 1$ and P''' of degree greater than $q - 1$ in z_2, \bar{z}_2 . Then theorem 3 implies $C(P') = 0$. Compute $c_0(f) + C(P''')$ and take the coefficient(s) of P'' in such a way that $c_0(f) + C(P'') + C(P''') \neq 0$. Given arbitrary, but fixed P''' , the latter inequality holds for generic P'' . This proves that $c_0(f) + C(P) \neq 0$ for generic P . \square

Lemmas 4–7 show that in our case study the conditions of theorem 1 and 2 are met for generic perturbation terms. Note that generic means open and dense with respect to the Zariski topology. More precisely, the preceding discussion is summarized in the following result.

Theorem 8. *Consider a three-cell feed-forward network with cell dynamics given by the three-parameter family*

$$F_{\lambda, \kappa, \varepsilon}(z_1, z_2) = f_{\lambda, \kappa}(z_2, z_1) + \varepsilon P(z_2, z_1),$$

where $f_{\lambda, \kappa}$ is given by (26). For $q > p > 0$, with p and q relatively prime, the dynamics of the third cell has a $p : q$ resonance point at

$$(\lambda, \kappa, \varepsilon) = \left(\sqrt{\frac{1}{2} \left(1 - \frac{p^2}{q^2} \right)}, \pm \sqrt{\frac{q^2 - p^2}{q^2 + p^2}}, 0 \right).$$

For generic perturbation terms P the dynamics of the third cell undergoes a generic Hopf–Neïmark–Sacker bifurcation at this point. The resonance tongues in $(\lambda, \kappa, \varepsilon)$ -space emanate from a cuspidal edge in the (λ, κ) plane given by

$$\lambda = \frac{1}{\sqrt{\kappa^2 + 1}}, \quad \varepsilon = 0.$$

The tongue boundaries exhibit the standard generic geometry, given by the local model (20). See also figure 3.

4. Conclusion and future work

We have explored a framework for detecting the occurrence of generic Hopf–Neïmark–Sacker bifurcations of families of perturbed forced oscillators. Applying this framework, we have been able to show that such bifurcations occur in a large class of linear feed-forward networks of coupled cells, each with the same dynamics. We have shown in detail how to apply our framework to concrete examples of such cell dynamics.

Our approach is based on fine-tuning existing normal form techniques, which, applied to concrete systems, provide the generic conditions for the occurrence of Hopf–Neïmark–Sacker bifurcations. Furthermore, this approach allowed us to show that these bifurcations give rise to resonance tongues with ‘standard’ geometry.

In [5] we have developed a method, based on Liapunov–Schmidt reduction and \mathbb{Z}_q equivariant singularity theory, to find period q resonance tongues in a Hopf bifurcation for dissipative maps. We recovered the standard non degenerate results, but also illustrated the method on the degenerate case ($q \geq 7$). We are currently analysing the resonance tongues of such mildly degenerate systems, which are (singular) hypersurfaces in four-dimensional parameter space.

We also plan to investigate the geometry of resonance tongues in families of cell dynamics exhibiting mildly degenerate Hopf–Neïmark–Sacker bifurcations. This phenomenon only occurs in systems depending on more than three parameters, so the analysis is typically more

complicated than in this paper, especially since the resonance tongues are (singular) surfaces in parameter space of dimension at least four.

Acknowledgments

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Appendix A. Versal unfolding of the linear part: Jordan normal form

In this appendix we show how to bring the linear part of the forced oscillator (13) into the Jordan normal form at the equilibrium $y = y_\mu$. By applying a sequence of transformations, as already described in section 2.2, we arrange that for all parameter values in the region of interest the dynamics of the third cell has an equilibrium $z = 0$ and that the linear part of this equilibrium is in the Jordan normal form. The following lemma describes this sequence of transformations.

Lemma 9. *Suppose the forced oscillator*

$$\dot{y} = g(y, \mu) + \varepsilon G(y, t, \mu) \quad (37)$$

satisfies the generic Hopf bifurcation condition (17), for μ near μ_0 , and the normal frequency at μ_0 is equal to $\omega_0 = \frac{p}{q}$. Then there is a transformation of the form

$$z = Z + \varepsilon \varphi(t, \varepsilon, \mu), \quad (38)$$

which is 2π -periodic in t , such that successive application of the transformations $Y = y - y_\mu$, cf (15), $Z = s_\mu^{-1}(Y)$, cf (21) and (38), brings the perturbed system (37) into the form

$$\dot{z} = (i\omega(\mu) + a(\mu))z + h(z, \mu) + \varepsilon H(z, t, \varepsilon, \mu), \quad (39)$$

where

1. $z \mapsto h(z, \mu)$ has zero 1-jet at $z = 0$, i.e. $h(0, \mu) = 0$, $d_z h(0, \mu) = 0$ and $h(z, \mu)$ depends linearly on g ;
2. $H(z, t, \varepsilon, \mu)$ is 2π -periodic in t , $H(0, t, \varepsilon, \mu) = 0$, and $H(z, t, \varepsilon, \mu)$ depends linearly on G (and also on g , though not necessarily linearly).

Remark. The proof of this lemma is a straightforward calculation. In fact, we will find an explicit expression for the transformation (38), which, in turn, will allow us to determine explicit expressions for the terms $h(z, \mu)$ and $H(z, t, \varepsilon, \mu)$ in (39). From these expressions we will then be able to conclude that if g is a polynomial of degree m in z, \bar{z} , then

1. $h(z, \mu)$ is a polynomial of degree m in z, \bar{z} ;
2. $H(z, t, 0, \mu) = K(z, t, \mu) - K(0, t, \mu) + P_{m-1}(z, t, \mu)$,

where K is given by

$$K(Z, t, \mu) = \frac{1}{\Delta} \left(\bar{\sigma} G(y_\mu + \sigma Z + \tau \bar{Z}, t, \mu) - \tau \overline{G(y_\mu + \sigma Z + \tau \bar{Z}, t, \mu)} \right), \quad (40)$$

where $P_{m-1}(z, t, \mu)$ is a polynomial of degree at most $m - 1$ in z, \bar{z} and where σ and τ are given by (22) and $\Delta = |\sigma|^2 - |\tau|^2$. In particular, $K(Z, t, \mu)$ is 2π -periodic in t . These explicit expressions will be needed in the proof of theorem 3 in appendix D.

Proof. The strategy of the proof is as follows. We first show that successive application of the transformations $Y = y - y_\mu$ and $Z = s_\mu^{-1}(Y)$ brings the linear part of the *unperturbed* system, i.e. (37) with $\varepsilon = 0$, into the Jordan normal form. Then we apply these two transformations to the *perturbed* system (37), which reintroduces a time dependent constant term. We show that a third transformation of the form (38) eliminates this constant, in other words, brings the system into the 2π -periodic form (39), which has a 2π -periodic orbit $z = 0$.

Recall that we first arrange that the equilibrium of the *unperturbed* system is at the origin by introducing a localized variable Y , defined near $0 \in \mathbb{C}$ by

$$y = y_\mu + Y,$$

where y_μ is the equilibrium given by (15). In this way the unperturbed dynamics $\dot{y} = g(y, \mu)$ is transformed into the form

$$\dot{Y} = L_\mu(Y) + g_1(Y, \mu), \quad (41)$$

where L_μ is the linear part of $Y \mapsto g(y_\mu + Y, \mu)$ at $Y = 0$. In particular, $g_1(Y, \mu)$ has no constant and linear terms in Y and \bar{Y} , i.e. $g_1(0, \mu) = 0$ and $d_Y g_1(0, \mu) = 0$.

The second transformation is the inverse of s_μ given by $s_\mu(Z) = \sigma Z + \tau \bar{Z}$, cf (21), where σ and τ are given by (22). Putting

$$Z = s_\mu^{-1}(Y) = \Delta^{-1}(\bar{\sigma} Y - \tau \bar{Y}),$$

with $\Delta = |\sigma|^2 - |\tau|^2$, brings the *unperturbed* system (41) into the Jordan normal form, i.e.

$$\dot{Z} = (i\omega(\mu) + a(\mu))Z + h(Z, \mu),$$

where

$$h(Z, \mu) = \Delta^{-1}(\bar{\sigma} g_1(\sigma Z + \tau \bar{Z}, \mu) - \tau \overline{g_1(\sigma Z + \tau \bar{Z}, \mu)}). \quad (42)$$

In particular, the two-jet of $Z \mapsto h(Z, \mu)$ vanishes at $Z = 0$ for μ near μ_0 .

We now apply the transformations $Y = y - y_\mu$ and $Z = s_\mu^{-1}(Y)$ to the *perturbed* system (37). In other words, we put

$$y = y_\mu + \sigma Z + \tau \bar{Z},$$

which brings (37) into the Jordan normal form

$$\dot{Z} = (i\omega + a)Z + h(Z, \mu) + \varepsilon K(Z, t, \mu),$$

where $h(Z, \mu)$ is given by (42) and $K(Z, t, \mu)$ by (40), i.e.

$$K(Z, t, \mu) = \Delta^{-1}(\bar{\sigma} G(y_\mu + \sigma Z + \tau \bar{Z}, t, \mu) - \tau \overline{G(y_\mu + \sigma Z + \tau \bar{Z}, t, \mu)}).$$

Denoting the time derivative of φ by $\dot{\varphi}$, and observing that the inverse of the third transformation (38) is given by $Z = z - \varepsilon\varphi(t, \varepsilon, \mu)$, we see that the transformed system is of the form

$$\dot{z} = \dot{Z} + \varepsilon\dot{\varphi} = \Phi(z, \varphi, t, \varepsilon, \mu) + \varepsilon\dot{\varphi}, \quad (43)$$

where

$$\Phi(z, \varphi, t, \varepsilon, \mu) = (i\omega + a)(z - \varepsilon\varphi) + h(z - \varepsilon\varphi, \mu) + \varepsilon K(z - \varepsilon\varphi, t, \mu).$$

Here we suppress the dependence of a and ω on μ and the dependence of φ and its derivative $\dot{\varphi}$ on t .

Our goal is to find a function $\varphi = \varphi(t, \varepsilon, \mu)$ such that the terms on the right-hand side independent of z and \bar{z} vanish, i.e. to reduce (43) to a system of the form $\dot{z} = O(|z|)$, uniformly

in (t, ε, μ) . Since (43) is of the form

$$\dot{z} = \Phi(0, \varphi, t, \varepsilon, \mu) + \varepsilon \dot{\varphi} + O(|z|),$$

we have to solve the differential equation

$$\Phi(0, \varphi, t, \varepsilon, \mu) + \varepsilon \dot{\varphi} = 0 \quad (44)$$

for $\varphi = \varphi(t, \varepsilon, \mu)$. Putting

$$K_0(t, \mu) = K(0, t, \mu) \quad \text{and} \quad K_1(\varphi, t, \varepsilon, \mu) = K(-\varepsilon\varphi, t, \mu) - K(0, t, \mu)$$

yields

$$\Phi(0, \varphi, t, \varepsilon, \mu) = -\varepsilon \left((i\omega + a)\varphi + K_0(t, \mu) \right) + h(-\varepsilon\varphi, \mu) + \varepsilon K_1(\varphi, t, \varepsilon, \mu).$$

Since the two-jet of $z \mapsto h(z, \mu)$ vanishes at $z = 0$, we have $h(-\varepsilon\varphi, \mu) = O(\varepsilon^2)$. Similarly, $K_1(\varphi, t, \varepsilon, \mu) = O(\varepsilon)$. Therefore, the division lemma guarantees the existence of a smooth function $K_2 : \mathbb{C} \times \mathbb{C} \times \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that

$$h(-\varepsilon\varphi, \mu) + \varepsilon K_1(\varphi, t, \varepsilon, \mu) = \varepsilon^2 K_2(\varphi, t, \varepsilon, \mu).$$

Note that also $K_2(\varphi, t, \varepsilon, \mu)$ is 2π -periodic in t . Therefore, (43) is of the form

$$\dot{z} = \varepsilon \left(\dot{\varphi} - (i\omega + a)\varphi + K_0(t, \mu) + \varepsilon K_2(\varphi, t, \varepsilon, \mu) \right) + O(|z|).$$

Observe that $e^{2\pi(i\omega+a)} \notin \mathbb{Z}$ for μ near μ_0 , since $a(\mu_0) = 0$ and $e^{2\pi i\omega(\mu_0)} \notin \mathbb{Z}$. Therefore, there is a unique function $\varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$, such that $\varphi(t, \varepsilon, \mu)$ is 2π -periodic in t , and

$$\dot{\varphi} - (i\omega + a)\varphi + K_0(t, \mu) + \varepsilon K_2(\varphi, t, \varepsilon, \mu) = 0,$$

and such that $\varphi_0(t, \mu) := \varphi(t, 0, \mu)$ is of the form

$$\varphi_0(t, \mu) = e^{(i\omega(\mu)+a(\mu))t} \left(c_0 - \int_0^t e^{-(i\omega(\mu)+a(\mu))s} K_0(s, \mu) ds \right), \quad (45)$$

with

$$c_0 = (e^{2\pi(i\omega(\mu)+a(\mu))} - 1)^{-1} \int_0^{2\pi} e^{-(i\omega(\mu)+a(\mu))s} K_0(s, \mu) ds.$$

With this choice of φ , transformation (38) reduces the system to one of the forms $\dot{z} = O(|z|)$. To prove that the reduced system is of the form (39) it remains to analyse the right-hand side of this system in more detail. To this end, (43) and (44) imply that the transformed system is of the form

$$\begin{aligned} \dot{z} &= \Phi(z, \varphi, t, \varepsilon, \mu) - \Phi(0, \varphi, t, \varepsilon, \mu) \\ &= (i\omega + a)z + h(z - \varepsilon\varphi, \mu) - h(-\varepsilon\varphi, \mu) + \varepsilon \left(K(z - \varepsilon\varphi, t, \mu) - K(-\varepsilon\varphi, t, \mu) \right). \end{aligned}$$

The Taylor expansion with respect to ε shows that

$$h(z - \varepsilon\varphi, \mu) - h(-\varepsilon\varphi, \mu) = h(z, \mu) + \varepsilon h_1(z, t, \varepsilon, \mu), \quad (46)$$

where $h_1 : \mathbb{C} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a smooth function of the form

$$h_1(z, t, \varepsilon, \mu) = -\varphi_0(t, \mu) h_z(z, \mu) - \overline{\varphi_0(t, \mu)} h_{\bar{z}}(z, \mu) + O(\varepsilon),$$

uniformly in (z, t, μ) , which is 2π -periodic in t . Taking $z = 0$ in (46) shows that

$$h_1(0, t, \varepsilon, \mu) \equiv 0. \quad (47)$$

Putting

$$H(z, t, \varepsilon, \mu) = h_1(z, t, \varepsilon, \mu) + K(z - \varepsilon\varphi, t, \varepsilon, \mu) - K(-\varepsilon\varphi, t, \varepsilon, \mu), \quad (48)$$

we see that the transformed system is of the form (39). The function $H(z, t, \varepsilon, \mu)$ is 2π -periodic in t , and, in view of (47), it satisfies $H(0, t, \varepsilon, \mu) = 0$. Furthermore, (48) and (40) show that $H(z, t, \varepsilon, \mu)$ depends on g and linearly G . This completes the proof of the lemma. \square

We have given the expressions for H in detail, since these will be used in the proof of theorem 3 to be presented in appendix D. In fact, we will need the following result.

Lemma 10. *For $\varepsilon = 0$, the function $H(z, t, \varepsilon, \mu)$, given by (48), satisfies*

$$H(z, t, 0, \mu_0) = -\varphi_0(t, \mu_0) h_z(z, \mu_0) - \overline{\varphi_0(t, \mu_0)} h_{\bar{z}}(z, \mu_0) + K(z, t, \mu_0) - K(0, t, \mu_0), \quad (49)$$

with

$$K(z, t, \mu_0) = \overline{\sigma_0} \Lambda_P(z, t) - \tau_0 \overline{\Lambda_P(z, t)}, \quad (50)$$

$$h(z, \mu_0) = \overline{\sigma_0} \lambda_f(z) - \tau_0 \overline{\lambda_f(z)} + \text{aff}(z, \bar{z}), \quad (51)$$

where $\text{aff}(z, \bar{z})$ is an affine function of z, \bar{z} , where $\Lambda_P(z, t)$ is defined by (25) and where

$$\lambda_f(z) = \Delta_0^{-1} f(y_0 + \sigma_0 z + \tau_0 \bar{z}). \quad (52)$$

Proof. Expressions (49) and (50) are straightforward consequences of (48) and (46). In view of the defining relations (42) and (14) of h in terms of g and of g in terms of f , respectively, identity (51) follows from the observation that

$$\begin{aligned} g_1(\sigma_0 z + \tau_0 \bar{z}, \mu_0) &= g(y_0 + \sigma_0 z + \tau_0 \bar{z}, \mu_0) - L_{\mu_0}(\sigma_0 z + \tau_0 \bar{z}) \\ &= f(y_0 + \sigma_0 z + \tau_0 \bar{z}, \mu_0) + \text{aff}(z, \bar{z}). \end{aligned} \quad \square$$

Appendix B. Derivation of normal forms

Methods for bringing the continuous dynamical system into the polynomial normal form are well known from the literature; See, e.g. [3, 27, 32]. In this appendix we show how to fine-tune this standard normal form procedure in such a way that the normal form provides additional information revealing the occurrence of generic Hopf–Neĭmark–Sacker bifurcations. In section B.1 we show that the terms of order i in the normal form depend linearly on the terms of order i in the original system and polynomially on the terms of lower order. Section B.2 presents the normal form procedure for periodic systems, and in section B.3 we use these normal form results to prove theorem 1. The results presented in this appendix are of independent interest, since their scope is much wider than the class of forced oscillators appearing in the dynamics of the third cell.

Appendix B.1. A normal form procedure

In this section we derive a normal form for the larger class of 2π -periodic systems of the form (39) where the system is in $p : q$ resonance, i.e. $\omega(\mu_0) = \frac{p}{q}$ for some Hopf parameter μ_0 , with p and q relatively prime. The normal form corresponds to the rotationally symmetric Hopf normal form for $\varepsilon = 0$. It turns out that the full normal form is not rotationally symmetric, but exhibits a \mathbb{Z}_q -symmetry.

Our approach to the computation of normal forms is a slight extension of the well-known methods introduced in [32]. This procedure transforms the terms of the vector field in ‘as simple a form as possible’, up to a user-defined order. It does so via iteration with respect to

the total degree of these terms. Before focusing on periodic systems in the plane we review the method for computing normal forms in a general Lie algebra setting and isolate some of its features that are crucial for our subsequent study of subharmonics in the forced oscillator (39).

Lie series expansion. For non negative integers m we denote the space of vector fields (on some Euclidean space of arbitrary dimension) of total degree m by \mathcal{H}_m and the space of vector fields with vanishing derivatives up to and including order m at $0 \in \mathbb{C}$ by \mathcal{F}_m . Note that $\mathcal{F}_m = \prod_{k \geq m} \mathcal{H}_k$.

Lemma 11. *Let X and Y be vector fields on \mathbb{C} , where X is of the form*

$$X = X^{(1)} + X^{(2)} + \dots + X^{(N)} \quad \text{mod } \mathcal{F}_{N+1}, \quad (53)$$

with $X^{(n)} \in \mathcal{H}_n$ and $Y \in \mathcal{H}_m$, with $m \geq 2$. Let Y_t , $t \in \mathbb{R}$, be the one-parameter group generated by Y and let $X^t = (Y_t)_(X)$. Then*

$$X^t = X + \sum_{k=1}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \text{ad}(Y)^k(X) \quad \text{mod } \mathcal{F}_{N+1} \quad (54)$$

$$= X + \sum_{n=1}^N \sum_{k=1}^{\lfloor \frac{N-n}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \text{ad}(Y)^k(X^{(n)}) \quad \text{mod } \mathcal{F}_{N+1}. \quad (55)$$

Proof. We follow the approach of Takens [32] and Broer *et al* [3,4] to obtain the Taylor series of X^t with respect to t in $t = 0$ using the basic identity

$$\frac{\partial}{\partial t} X^t = [X^t, Y] = -\text{ad}(Y)(X^t).$$

Using this relation, we inductively prove that

$$\frac{\partial^k}{\partial t^k} X^t = (-1)^k \text{ad}(Y)^k(X^t).$$

Using the latter identity for $t = 0$, we obtain the formal Taylor series

$$\begin{aligned} X^t &= \sum_{k \geq 0} \frac{1}{k!} t^k \left. \frac{\partial^k}{\partial t^k} \right|_{t=0} X^t \\ &= \sum_{k \geq 0} \frac{(-1)^k}{k!} t^k \text{ad}(Y)^k(X). \end{aligned}$$

Since $Y \in \mathcal{H}_m$, the operator $\text{ad}(Y)^k$ increases the degree of each term in its argument by $k(m-1)$. Since the terms of lowest order in X are linear, we see that

$$\text{ad}(Y)^k(X) = 0 \quad \text{mod } \mathcal{F}_{N+1},$$

if $1 + k(m-1) > N$. Therefore,

$$\begin{aligned} X^t &= \sum_{k=0}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \text{ad}(Y)^k(X) \quad \text{mod } \mathcal{F}_{N+1} \\ &= X + \sum_{k=1}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \text{ad}(Y)^k(X) \quad \text{mod } \mathcal{F}_{N+1}, \end{aligned}$$

which proves (54). In view of (53) the latter identity expands to

$$X^t = \sum_{n=1}^N \sum_{k=0}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} t^k \operatorname{ad}(Y)^k(X^{(n)}) \quad \text{mod } \mathcal{F}_{N+1}. \quad (56)$$

Since $\operatorname{ad}(Y)^k(X^{(n)}) \in \mathcal{H}_{n+k(m-1)}$, we see that

$$\operatorname{ad}(Y)^k(X^{(n)}) = 0 \quad \text{mod } \mathcal{F}_{N+1},$$

for $k > \frac{N-n}{m-1}$. Therefore, for fixed index n , the inner sum in (56) can be truncated at $k = \lfloor \frac{N-n}{m-1} \rfloor$, which concludes the proof of (55). \square

Computing normal forms. Consider a vector field X having a singular point with semisimple linear part S . Our goal is to bring X into the normal form to some prescribed order N . Since S is semisimple we know that

$$\mathcal{H}_m = \operatorname{Ker} \operatorname{ad}(S) + \operatorname{Im} \operatorname{ad}(S).$$

Here $\operatorname{ad}(S) : \mathcal{H}_m \rightarrow \mathcal{H}_m$ is the restriction to \mathcal{H}_m of the adjoint action of the linear part S of X . The linear operators $G_m : \mathcal{H}_m \rightarrow \mathcal{H}_m$ and $B_m : \mathcal{H}_m \rightarrow \mathcal{H}_m$ are the corresponding projections onto $\operatorname{Ker} \operatorname{ad}(S)$ and $\operatorname{Im} \operatorname{ad}(S)$, respectively. So, $X^{(m)} = G_m(X^{(m)}) + B_m(X^{(m)})$, for $X^{(m)} \in \mathcal{H}_m$.

Lemma 12 (normal form lemma [32]). *Consider a vector field $X = S + X^{(2)} + \dots + X^{(m)} \bmod \mathcal{F}_{m+1}$, with $X^{(i)} \in \mathcal{H}_i$. There is a transformation Φ bringing the vector field X into the normal form*

$$\Phi_*(X) = S + G^{(2)} + \dots + G^{(m)} \quad \text{mod } \mathcal{F}_{m+1},$$

for any $m \geq 2$, where $G^{(i)} \in \mathcal{H}_i$ belongs to $\operatorname{Ker} \operatorname{ad}(S)$. Furthermore, $G^{(i)}$ is of the form

$$G^{(i)} = G_i(X^{(i)}) + \Gamma_i(S, X^{(2)}, \dots, X^{(i-1)}),$$

where $\Gamma_i(S, X^{(2)}, \dots, X^{(i-1)})$ is a polynomial in the coefficients of $S, X^{(2)}, \dots, X^{(i-1)}$.

The first part of the lemma is well known, cf [32]. The second part follows from a careful analysis of the normal form transformation, which we give now.

Proof. The normal form is determined in an iterative process, which successively brings the terms of order $2, \dots, N$ into the normal form. So assume that X has been brought into the form

$$\tilde{X} = S + G^{(2)} + \dots + G^{(m-1)} + \tilde{X}^{(m)} + \dots + \tilde{X}^{(N)} \quad \text{mod } \mathcal{F}_{N+1}, \quad (57)$$

where the homogeneous parts of \tilde{X} satisfy the following conditions.

$P_1(m)$: $G^{(i)} \in \mathcal{H}_i$ belongs to $\operatorname{Ker} \operatorname{ad}(S)$, for $2 \leq i < m$, and is of the form

$$G^{(i)} = G_i(X^{(i)}) + \Gamma_i, \quad \text{where } \Gamma_i \text{ depends polynomially on the coefficients of } S, X^{(2)}, \dots, X^{(i-1)}.$$

$P_2(m)$: $\tilde{X}^{(i)} \in \mathcal{H}_i$ is of the form $\tilde{X}^{(i)} = X^{(i)} + \Xi_i$, for $m \leq i \leq N$, where Ξ_i depends polynomially on the coefficients of $S, X^{(2)}, \dots, X^{(i-1)}$;

Note that (57) is trivially true for $m = 1$, since the linear part is already in the normal form. We determine a normal form transformation bringing the vector field into the form (57) with m replaced by $m + 1$, such that $P_1(m + 1)$ and $P_2(m + 1)$ hold for this incremented value of m . By induction, this will prove the lemma.

The normal form transformation will be the time-1-map of a vector field $Y \in \mathcal{H}_m$, to be determined in such a way that the term of order m of the transformed vector field is in the

normal form. Let Y_t , $t \in \mathbb{R}$, be the one-parameter group generated by Y and let $X^t = (Y_t)_*(\tilde{X})$. To determine the homogeneous components of X^1 , we apply lemma 11, replacing X with \tilde{X} and taking into account that $\tilde{X}^{(1)} = S$ and $\tilde{X}^{(i)} = G^{(i)}$, for $2 \leq i < m$. Splitting off the term for $n = 1$ in (55) we see that

$$\begin{aligned} X^1 = \tilde{X} - \text{ad}(Y)(S) + \sum_{k=2}^{\lfloor \frac{N-1}{m-1} \rfloor} \frac{(-1)^k}{k!} \text{ad}(Y)^k(S) \\ + \sum_{n=2}^N \sum_{k=1}^{\lfloor \frac{N-n}{m-1} \rfloor} \frac{(-1)^k}{k!} \text{ad}(Y)^k(\tilde{X}^{(n)}) \quad \text{mod } \mathcal{F}_{N+1}. \end{aligned} \quad (58)$$

Recall that $\tilde{X}^{(m)} = G^{(m)} + B^{(m)}$, with $G^{(m)} = G_m(\tilde{X}^{(m)}) \in \text{Ker ad}(S)$ and $B^{(m)} = B_m(X^{(m)}) \in \text{Im ad}(S)$. Now there is a unique $Y \in \text{Im ad}(S)$ satisfying the *homological equation*

$$\text{ad}(S)(Y) = -B^{(m)}. \quad (59)$$

Since $\tilde{X} - \text{ad}(Y)(S) = \tilde{X} + \text{ad}(S)(Y) = \tilde{X} - B^{(m)} = S + G^{(2)} + \dots + G^{(m)} \text{ mod } \mathcal{F}_{m+1}$, identities (58) and (59) imply that X^1 is in the normal form to order m . To see this, observe that $\text{ad}(Y)^k(\tilde{X}^{(n)}) \in \mathcal{H}_{n+k(m-1)}$. Therefore, both sums on the right-hand side of (58) consist of terms in \mathcal{F}_{m+1} . In other words,

$$X^1 = S + G^{(2)} + \dots + G^{(m-1)} + G^{(m)} \quad \text{mod } \mathcal{F}_{m+1}.$$

Observe that $G^{(m)} = G_m(\tilde{X}^{(m)}) + \Gamma_m$, where $\Gamma_m = G_m(\Xi_m)$. Since Ξ_m is a polynomial in $X^{(2)}, \dots, X^{(m)}$, the same thing holds for Γ_m . This proves $P_1(m+1)$.

To prove $P_2(m+1)$, observe that the solution Y of the homological equation (59) depends linearly on the coefficients of $\tilde{X}^{(m)}$ and, hence, in view of $P_2(m)$, polynomially on the coefficients of $S, X^{(2)}, \dots, X^{(m)}$. Therefore, each of the terms $\text{ad}(Y)^k(\tilde{X}^{(n)})$ in (58) depends polynomially on these coefficients. In other words, the homogeneous term of order n , $m < n \leq N$, of the transformed vector field X^1 is obtained by adding a finite number of polynomials in the coefficients of $S, X^{(2)}, \dots, X^{(m)}$ to $\tilde{X}^{(n)}$. Together with $P_2(m)$ this observation implies $P_2(m+1)$ and, hence, completes the proof of the normal form lemma. \square

Appendix B.2. Normal form of periodic systems

The Lie algebra of planar periodic systems. When dealing with real 2π -periodic vector fields on \mathbb{R}^2 we identify \mathbb{R}^2 with \mathbb{C} , by associating the point (x_1, x_2) in \mathbb{R}^2 with $x_1 + ix_2$ in \mathbb{C} . The real 2π -periodic vector fields correspond to systems on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ of the form

$$X = X_{\mathbb{R}} \frac{\partial}{\partial z} + \overline{X_{\mathbb{R}}} \frac{\partial}{\partial \bar{z}} + X_{\mathbb{S}} \frac{\partial}{\partial t}, \quad (60)$$

where $X_{\mathbb{R}} : \mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z}) \rightarrow \mathbb{C}$ and $X_{\mathbb{S}} : \mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z}) \rightarrow \mathbb{C}$ are 2π -periodic functions, i.e. $X_{\mathbb{R}}(z, t + 2\pi) = X_{\mathbb{R}}(z, t)$ and $X_{\mathbb{S}}(z, t + 2\pi) = X_{\mathbb{S}}(z, t)$. Recalling that

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right),$$

we see that the real vector field X , defined on $\mathbb{R}^2 \times (\mathbb{R}/2\pi\mathbb{Z})$ by

$$X = Y_1 \frac{\partial}{\partial x_1} + Y_2 \frac{\partial}{\partial x_2} + Y_3 \frac{\partial}{\partial t},$$

corresponds to the vector field

$$X = Y \frac{\partial}{\partial z} + \bar{Y} \frac{\partial}{\partial \bar{z}} + Y_3 \frac{\partial}{\partial t},$$

on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$, where $Y = Y_1 + iY_2$. These vector fields form a Lie subalgebra of the algebra of vector fields on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$. This is justified by the fact that the Lie bracket of two real 2π -periodic vector fields X and Y on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ is given by

$$[X, Y] = \langle X, Y \rangle_{\mathbb{R}} \frac{\partial}{\partial z} + \overline{\langle X, Y \rangle_{\mathbb{R}}} \frac{\partial}{\partial \bar{z}} + \langle X, Y \rangle_{\mathbb{S}} \frac{\partial}{\partial t}, \quad (61)$$

where the \mathbb{R} -bilinear antisymmetric forms $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{S}}$ are defined by

$$\langle X, Y \rangle_{\mathbb{R}} = X(Y_{\mathbb{R}}) - Y(X_{\mathbb{R}}) \quad \text{and} \quad \langle X, Y \rangle_{\mathbb{S}} = X(Y_{\mathbb{S}}) - Y(X_{\mathbb{S}}). \quad (62)$$

The proof consists of a straightforward computation. Therefore, we omit the details.

The adjoint action in a periodic context. Let \mathcal{H}_m , $m \geq 1$, be the space of *real* vector fields on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ with *vanishing* $\frac{\partial}{\partial t}$ component, cf (60), whose real components $X_{\mathbb{R}}(z, t)$ are homogeneous polynomials of degree m in (z, \bar{z}) with 2π -periodic coefficients. In other words,

$$X_{\mathbb{R}}(z, t) = \sum_{k=0}^m f_k(t) z^k \bar{z}^{m-k}, \quad (63)$$

where f_k is 2π -periodic. The Fourier coefficients of a 2π -periodic function f are denoted by $c_n(f)$, i.e. for $n \in \mathbb{Z}$:

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} f(s) ds.$$

To extend the context of Hopf bifurcations we consider 2π -periodic systems with the linear part of the form

$$S = i\omega \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \frac{\partial}{\partial t}, \quad (64)$$

where ω is a real number. The following result describes the action of $\text{ad}(S)$ on the space of 2π -periodic vector fields.

Lemma 13 (adjoint action $\text{ad}(S)$). *The operator $\text{ad}(S)$, associated with (64), leaves the subspace of vector fields with zero $\frac{\partial}{\partial t}$ -component invariant. The restriction of this operator to \mathcal{H}_m is semisimple, i.e. for $m \geq 2$ there is a decomposition*

$$\mathcal{H}_m = \text{Ker ad}(S) + \text{Im ad}(S). \quad (65)$$

If $G_m : \mathcal{H}_m \rightarrow \mathcal{H}_m$ is the corresponding projection operator onto $\text{Ker ad}(S)$, then, for a vector field X with $X_{\mathbb{R}}$ of the form (63), the image $G_m(X)$ has a $\frac{\partial}{\partial z}$ -component of the form

$$G_m(X)_{\mathbb{R}} = \sum_k c_{(m+1-2k)\omega}(f_k) e^{i(2k-m-1)\omega t} z^k \bar{z}^{m-k}, \quad (66)$$

where k ranges over the index set $\{k \mid 0 \leq k \leq m \text{ and } (2k - m - 1)\omega \in \mathbb{Z}\}$.

Proof. If $X_{\mathbb{S}} = 0$, then (62) implies $\langle S, X \rangle_{\mathbb{S}} = 0$, so $\text{ad}(S)$ leaves the subspace of vector fields with a zero $\frac{\partial}{\partial t}$ -component invariant. Furthermore, a straightforward computation shows

$$\text{ad}(S)X_{\mathbb{R}} = \sum_{k=0}^m (i\omega(2k - m - 1) f_k(t) + f'_k(t)) z^k \bar{z}^{m-k}.$$

Table 1. The basis for the kernel of $\text{ad}(S) : \mathcal{H}_m \rightarrow \mathcal{H}_m$, for $0 \leq m \leq q-1$, in the case $\omega = \frac{p}{q}$, and the projection $G_m(X) \in \text{Ker ad}(S)$, where $X_{\mathbb{R}}$ is of the form (63).

	Basis of $\text{Ker ad}(S)$	$G_m(X)_{\mathbb{R}}$
$0 \leq m < q-1, m \text{ even}$	\emptyset	0
$0 \leq m < q-1, m = 2n+1$	$\{Y^0\}$, with $Y_{\mathbb{R}}^0 = z z ^{2n}$	$c_0(f_{n+1}) Y_{\mathbb{R}}^0$
$m = q-1, m \text{ even}$	$\{Y^1\}$, with $Y_{\mathbb{R}}^1 = \bar{z}^{q-1} e^{ipt}$	$c_{-p}(f_{q-1}) Y_{\mathbb{R}}^1$
$m = q-1 = 2n+1$	$\{Y^0, Y^1\}$, with $Y_{\mathbb{R}}^0 = z z ^{2n}$ and $Y_{\mathbb{R}}^1 = \bar{z}^{q-1} e^{ipt}$	$c_0(f_{n+1}) Y_{\mathbb{R}}^0$ $+ c_{-p}(f_{q-1}) Y_{\mathbb{R}}^1$

Therefore, $\text{ad}(S)X = 0$ iff $i\omega(2k-m-1)f_k(t) + f'_k(t) = 0$, for $0 \leq k \leq m$. If $\omega(2k-m-1) \notin \mathbb{Z}$, the only 2π -periodic solution is $f_k = 0$. If $\omega(2k-m-1) \in \mathbb{Z}$, then any 2π -periodic solution is of the form $f_k(t) = f_k(0)e^{i\omega(m+1-2k)t}$. Therefore, the kernel of $\text{ad}(S)$ consists of all vector fields X with the real part $X_{\mathbb{R}}$ belonging to the space

$$\text{Span} \{e^{i\omega(m+1-2k)t} z^k \bar{z}^{m-k} \mid 0 \leq k \leq m \quad \text{and} \quad (2k-m-1)\omega \in \mathbb{Z}\}.$$

Let $G_m : \mathcal{H}_m \rightarrow \mathcal{H}_m$ be the projection of X onto $\text{Ker ad}(S)$, defined by (66). A straightforward computation shows that $X - G_m(X) \in \text{Ker ad}(S)$. Therefore, G_m gives rise to the splitting (65), so $\text{ad}(S)$ is semisimple. \square

Remark. In our study of Hopf–Neĭmark–Sacker bifurcations, we will focus on periodic vector fields with $\omega = \frac{p}{q}$. Using (66) we see that in this case the term without rotational symmetry of lowest order is of degree $q-1$ in z, \bar{z} . More precisely, Table 1 presents the kernel of $\text{ad}(S)$ and the corresponding projection (66) for $m < q$. This result will be used in the next section to derive the normal form to order q . Note that in [15] we obtained normal forms for periodic systems by averaging over time. The two approaches are closely related, as explained in [3].

Appendix B.3. Normal form to order q : proof of theorem 1

The normal form for parametrized planar periodic systems of the form (39) can be determined using the methods of section B.1. More precisely, we consider a parametrized 2π -periodic system X with $X_{\mathbb{R}}$ of the form

$$X_{\mathbb{R}}(z, t, \mu) = (i\omega(\mu) + a(\mu))z + h(z, \mu) + \varepsilon H(z, t, \mu), \quad (67)$$

where $\mu \in \mathbb{R}^k$ and ε is a small real parameter, such that, for $\mu_0 \in \mathbb{R}^k$,

1. $a(\mu) \in \mathbb{R}$ and $\omega(\mu) \in \mathbb{R}$ satisfy $a(\mu_0) = 0$ and $\omega_N := \omega(\mu_0) = \frac{p}{q}$, with p and q relatively prime;
2. h and H contain no terms independent of z and \bar{z} (i.e. $h(0, \mu) = 0$ and $H(0, t, \mu) = 0$);
3. h does not contain terms that are linear in z and \bar{z} .

Lemma 14 (normal form to order q). *The parametrized system (67), satisfying properties 1, 2 and 3, has the normal form*

$$\dot{z} = i\omega_N z + (\alpha(\mu, \varepsilon) + i\delta(\mu, \varepsilon))z + zF(|z|^2, \mu, \varepsilon) + \varepsilon \gamma(\mu, \varepsilon) \bar{z}^{q-1} e^{ipt} + O(|z|^q), \quad (68)$$

uniformly in (ε, μ) , where the $O(|z|^q)$ terms are 2π -periodic in t . Here $F(|z|^2, \mu, \varepsilon)$ is a complex polynomial of degree at most $q-2$ with $F(0, \mu, \varepsilon) = 0$. Furthermore,

- α and δ are real-valued functions such that $\alpha(\mu, 0) = a(\mu)$ and $\delta(\mu, 0) = \omega(\mu) - \omega(\mu_0)$. In particular, α and δ vanish at $(\mu, \varepsilon) = (\mu_0, 0)$.

- $\gamma(\mu, \varepsilon) \in \mathbb{C}$, and

$$\gamma(\mu_0, 0) = d + \Psi_h,$$

where d is the coefficient of the term $\varepsilon \bar{z}^{q-1} e^{ipt}$ in the Taylor–Fourier expansion of $H(z, t, \mu_0)$ at $z = 0$ and Ψ_h is a polynomial in the coefficients of the $(q - 1)$ st Taylor polynomial of $h(\cdot, \mu_0)$ in z, \bar{z} at $z = 0$.

Proof. To separate the contribution of h and H to the normal form and to the normal form transformation, we regard the normal form procedure as consisting of two successive steps.

Step 1. Determine the normal form to order q of the unperturbed system, i.e. of system (67) with $\varepsilon = 0$, and apply the corresponding normal form transformation to the complete (perturbed) system (67). Since the unperturbed system is time independent, it follows from lemma 13 that this first step yields an intermediate vector field of the form

$$\dot{z} = (i\omega(\mu) + a(\mu))z + z\tilde{h}(|z|^2, \mu) + O(|z|^q), \quad (69)$$

uniformly in μ , where $\tilde{h}(|z|^2, \mu)$ is a time-independent polynomial of degree less than q in z and \bar{z} . Indeed, the normal form terms of degree m belong to the kernel of $\text{ad}(S) : \mathcal{H}_m \rightarrow \mathcal{H}_m$, cf lemma 12. According to lemma 13, the time independent terms in this kernel are 0, if m is odd, and multiples of the vector field Y with $Y_{\mathbb{R}} = z|z|^{2k}$, if $m = 2k + 1$. This proves (69).

Step 2. Applying the normal form procedure of step 1 to the full system (67) yields a system of the form

$$\dot{z} = (i\omega(\mu) + a(\mu))z + z\tilde{h}(|z|^2, \mu) + \varepsilon \tilde{H}(z, t, \mu) + O(|z|^q), \quad (70)$$

uniformly in (t, μ, ε) . Since $\omega(\mu_0) = \frac{p}{q}$, we apply the normal form procedure presented in section B.1, where the linear part is given by

$$S = i\omega_N \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \frac{\partial}{\partial t},$$

with $\omega_N = \omega(\mu_0) = \frac{p}{q}$. For $2 \leq m \leq q - 1$, the kernel of $\text{ad}(S) : \mathcal{H}_m \rightarrow \mathcal{H}_m$ is given in table 1. Therefore, the normal form of X is of the form (68). Since this normal form coincides with (69) for $\varepsilon = 0$, it follows that $\delta(\mu, 0) = \omega(\mu) - \omega_N$ and $\alpha(\mu, 0) = a(\mu)$.

To prove that $\gamma(\mu_0, 0) = d + \Psi_h$, we consider the system, given by (70) for $\mu = \mu_0$, and derive its normal form. Then $\gamma(\mu_0, 0)$ is the coefficient of the term $\varepsilon \bar{z}^{q-1} e^{ipt}$ in this normal form.

For $\mu = \mu_0$ system (69) has the rotationally symmetric form

$$X_0 = S + \sum_{n=2}^{q-1} G_0^{(n)} + O(|z|^q),$$

where

$$G_0^{(n)} = \begin{cases} 0, & \text{if } n \text{ is even,} \\ a_k z^{k+1} \bar{z}^k, & \text{if } n = 2k + 1. \end{cases} \quad (71)$$

Here the complex number a_k is a constant. Let X be system (70) for $\mu = \mu_0$, then X is of the form

$$X = X_0 + \varepsilon \sum_{n=1}^{q-1} X^{(n)} + O(|z|^q + \varepsilon^2),$$

uniformly in t . Here $X^{(n)}$ is homogeneous of degree n in z, \bar{z} and 2π -periodic in t , but independent of the parameters μ and ε . Since the normal form transformation of step 1 is a

near identity transformation of the form $z \mapsto z + O(|z|^2)$, where the higher order terms depend polynomially on the coefficients of the $(q-1)$ -jet of h at $z=0$, it follows that the coefficient of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in X is of the form $d + \Psi_h$, with d and Ψ_h as in the statement of the lemma. We shall prove that further normal form transformations leave this coefficient invariant.

Again we derive the normal form of X inductively with respect to the degree of the terms in z, \bar{z} . To this end we introduce the following predicate, for $0 \leq m \leq q$:

$P(m)$: there is a transformation Φ_m on $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ bringing X into the form

$$(\Phi_m)_*(X) = X_0 + \varepsilon \sum_{n=1}^{m-1} G_1^{(n)} + \varepsilon \sum_{n=m}^{q-1} \tilde{X}^{(n)} + O(|z|^q + \varepsilon^2), \quad (72)$$

where $G_1^{(n)}$ belongs to the kernel of $\text{ad}(S) : \mathcal{H}_n \rightarrow \mathcal{H}_n$. Furthermore, the coefficients of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in X and in $(\Phi_m)_*(X)$ are equal.

To see that $P(q)$ implies the statement of the lemma, recall that the kernel of $\text{ad}(S)$ is given by table 1. Therefore, the $\frac{\partial}{\partial z}$ -component of $(\Phi_q)_*(X)$ is of the form

$$\dot{z} = i\omega_N z + \sum_{k \geq 1, 2k+2 \leq q} (a_k + \varepsilon b_k) z |z|^{2k} + \varepsilon (d + \Psi_h) \bar{z}^{q-1} e^{ipt} + O(|z|^q + \varepsilon^2),$$

which implies the statement of the lemma.

We now prove inductively that $P(m)$ holds. Taking Φ_0 equal to the identity map we see that $P(0)$ holds. So assume $P(m)$ holds, for $0 \leq m < q$. Denote $(\Phi_m)_*(X)$ by \tilde{X} . As in the proof of lemma 12, we find the transformation bringing \tilde{X} into the form (72), with m replaced with $m+1$, by solving the homological equation

$$\text{ad}(S)(Y) = -\varepsilon B^{(m)}, \quad (73)$$

where the right-hand side is the component of $\varepsilon \tilde{X}^{(m)}$ in the image of $\text{ad}(S)$, up to sign, i.e. $\tilde{X}^{(m)} = G_1^{(m)} + B^{(m)}$. Then the degree m term of $(Y_1)_*(X)$ is equal to $G_0^{(m)} + \varepsilon G_1^{(m)}$, which is in the normal form.

The terms in the transformed vector field $(Y_1)_*(X)$ of order greater than m are determined using the Lie series expansion (55). Since (73) implies $Y = O(\varepsilon)$, it follows that $\text{ad}(Y)^k(\tilde{X}^{(n)}) = O(\varepsilon^k)$. Similarly, $\text{ad}(Y)(\varepsilon G_1^{(n)}) = O(\varepsilon^2)$, for $1 \leq n < m$, and $\text{ad}(Y)(\varepsilon \tilde{X}^{(n)}) = O(\varepsilon^2)$, for $m \leq n < q$. Putting $Y = \varepsilon \tilde{Y}$, we see that the transformed vector field is of the form

$$\begin{aligned} (Y_1)_*(\tilde{X}) &= \tilde{X} - \text{ad}(Y)(S) + \varepsilon \sum_{n=2}^{q-1} \text{ad}(\tilde{Y})(G_0^{(n)}) \\ &= X_0 + \varepsilon \sum_{n=1}^{m-1} G_1^{(n)} + \varepsilon (\tilde{X}^{(m)} + \text{ad}(\tilde{Y})(S)) \\ &\quad + \varepsilon \left(\sum_{n=m+1}^{q-1} \tilde{X}^{(n)} + \sum_{n=2}^{q-1} \text{ad}(\tilde{Y})(G_0^{(n)}) \right) + O(|z|^q + \varepsilon^2) \\ &= X_0 + \varepsilon \sum_{n=1}^m G_1^{(n)} + \varepsilon \sum_{n=m+1}^{q-1} \tilde{\tilde{X}}^{(n)} + O(|z|^q + \varepsilon^2), \end{aligned}$$

where

$$\tilde{\tilde{X}}^{(n)} = \tilde{X}^{(n)} + \text{ad}(\tilde{Y})(G_0^{(n-m+1)}). \quad (74)$$

Taking $\Phi_{m+1} = Y_1 \circ \Phi_m$ we see that the first part of $P(m)$ holds. To prove that the coefficients of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in $(Y_1)_*(\tilde{X})$ and in X are equal, we consider two cases.

Case 1. $m < q - 1$. Identity (74) implies that $\tilde{X}^{(q-1)} = \tilde{X}^{(q-1)} + \text{ad}(\tilde{Y})(G_0^{(q-m)})$. The normal form term is zero if $q - m$ is even, otherwise it contains a factor z^2 , since $q - m \geq 3$, cf (71). In the latter case $\text{ad}(\tilde{Y})(G_0^{(q-m)})$ contains a factor z , so the coefficients of $\bar{z}^{q-1} e^{ipt}$ in $\tilde{X}^{(q-1)}$ and $\tilde{X}^{(q-1)}$ are equal. Therefore, the transformation Y_1 leaves the coefficient of $\varepsilon \bar{z}^{q-1} e^{ipt}$ invariant.

Case 2. $m = q - 1$. Now the coefficient of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in $(Y_1)_*(X)$ is equal to the coefficient of $\bar{z}^{q-1} e^{ipt}$ in $G^{(q-1)}$. Now $G^{(q-1)} = G_{q-1}(\tilde{X}^{(q-1)})$, so in view of table 1 the coefficient of $\bar{z}^{q-1} e^{ipt}$ in $G^{(q-1)}$ is equal to the coefficient of $\bar{z}^{q-1} e^{ipt}$ in $G^{(q-1)}$ in $\tilde{X}^{(q-1)}$. According to the induction hypothesis $P(q - 1)$, the latter coefficient is equal to the coefficient of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in X . This concludes the proof of lemma 14. \square

Remarks. 1. It is not hard to prove that Ψ_h is a non-constant polynomial, which is identically 0 if h is rotationally symmetric.

2. Note that the hyperplane $\varepsilon = 0$ is mapped onto the hyperplane $\tilde{\varepsilon} = 0$ in $(\tilde{\mu}, \tilde{\varepsilon})$ -space. Furthermore, if $\mu_0 = (\mu_1^0, \dots, \mu_k^0)$, then the bifurcation point $(\mu, \varepsilon) = (\mu_0, 0)$ corresponds to $(\tilde{\mu}, \tilde{\varepsilon}) = (0, 0, \mu_3^0, \dots, \mu_k^0, 0)$.

3. We continue our study of the birth or death of subharmonics of order q by analysing the normal form (18) further. In the following we will drop the tildes from our notation.

Appendix C. Geometry of resonance tongues: proof of theorem 2

Existence of $2\pi q$ -periodic orbits: the Van der Pol transformation. Subharmonics of order q of the 2π -periodic forced oscillator (19) correspond to q -periodic orbits of the Poincaré time 2π -map $P : \mathbb{C} \rightarrow \mathbb{C}$. These periodic orbits of the Poincaré map are brought into one-to-one correspondence with the zeros of a vector field on a q -sheeted cover of the phase space $\mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z})$ via the *Van der Pol transformation*, cf [15, 24]. This transformation corresponds to a q -sheeted covering

$$\begin{aligned} \Pi : \mathbb{C} \times \mathbb{R}/(2\pi q\mathbb{Z}) &\rightarrow \mathbb{C} \times \mathbb{R}/(2\pi\mathbb{Z}), \\ (z, t) &\mapsto (ze^{itp/q}, t \pmod{2\pi\mathbb{Z}}) \end{aligned} \quad (75)$$

with cyclic Deck group of order q generated by

$$(z, t) \mapsto (ze^{2\pi ip/q}, t - 2\pi).$$

The Van der Pol transformation $\zeta = ze^{-i\omega_N t}$ lifts the forced oscillator (67) to the system

$$\dot{\zeta} = (\alpha + i\delta)\zeta + \zeta h(\zeta e^{i\omega_N t}, \mu) + \varepsilon H(\zeta e^{i\omega_N t}, t, \mu) \quad (76)$$

on the covering space $\mathbb{C} \times \mathbb{R}/(2\pi q\mathbb{Z})$. The latter system is \mathbb{Z}_q -equivariant. A straightforward application of (76) to the normal form (18) yields the following normal form for the lifted forced oscillator.

Lemma 15 (equivariant normal form of order q). *On the covering space, the lifted forced oscillator has the \mathbb{Z}_q -equivariant normal form:*

$$\dot{\zeta} = (\alpha + i\delta)\zeta + \zeta F(|\zeta|^2, \mu, \varepsilon) + \varepsilon \bar{\zeta}^{q-1} + O(|\zeta|^q), \quad (77)$$

where the $O(|\zeta|^q)$ terms are $2\pi q$ -periodic.

Resonance tongues for families of forced oscillators. Bifurcations of q -periodic orbits of the Poincaré map P on the base space correspond to bifurcations of fixed points of the Poincaré map \tilde{P} on the q -sheeted covering space introduced in connection with the Van der Pol transformation (75). Denoting the normal form system (68) on the base space by \mathcal{N} and the normal form system (77) of the lifted forced oscillator by $\tilde{\mathcal{N}}$, we see that

$$\Pi_*\tilde{\mathcal{N}} = \mathcal{N}.$$

The Poincaré mapping \tilde{P} of the normal form on the covering space now is the $2\pi q$ -period mapping

$$\tilde{P} = \tilde{\mathcal{N}}^{2\pi q} + O(|z|^q),$$

where $\tilde{\mathcal{N}}^{2\pi q}$ denotes the $2\pi q$ -map of the (planar) vector field $\tilde{\mathcal{N}}$. Following the corollary to the normal form theorem of [15, p 12], we conclude for the original Poincaré map P of the vector field X on the base space that

$$P = R_{2\pi\omega_N} \circ \tilde{\mathcal{N}}^{2\pi} + O(|z|^q),$$

where $R_{2\pi\omega_N}$ is the rotation over $2\pi\omega_N = 2\pi p/q$, which precisely is the Takens normal form [33] of P at $(z, \mu) = (0, 0)$.

Our interest is in the q -periodic points of P_μ , which correspond to the fixed points of \tilde{P}_μ . From now on we assume that our parameter space is three-dimensional and that $\mu = (\alpha, \delta, \varepsilon)$. Lemma 15 implies that the fixed point set of \tilde{P}_μ and the boundary thereof in the parameter space is approximately described by the discriminant set of

$$(a + i\delta)\zeta + \zeta F(|\zeta|^2, \mu) + \varepsilon \bar{\zeta}^{q-1}, \quad (78)$$

which is the truncated right-hand side of (77). This gives rise to the bifurcation equation that determines the boundaries of the resonance tongues. The following theorem implies that, under the conditions that $F_u(0, 0) \neq 0$, the order of tangency at the tongue tips is generic. Here $F_u(0, 0)$ is the partial derivative of $F(u, \mu)$ with respect to u .

Lemma 16 (bifurcation equations modulo contact equivalence). *Assume that $F_u(0, 0) \neq 0$. Then the polynomial (78) is \mathbb{Z}_q -equivariantly contact equivalent with the polynomial*

$$G(\zeta, \mu) = (a + i\delta + |\zeta|^2)\zeta + \varepsilon \bar{\zeta}^{q-1}. \quad (79)$$

The discriminant set of the polynomial $G(\zeta, \mu)$ is of the form

$$\delta = \pm \varepsilon (-a)^{(q-2)/2} + O(\varepsilon^2). \quad (80)$$

For a definition and similar use of contact equivalence we refer to [5]. Here it is sufficient to keep in mind that contact equivalent families have diffeomorphic (parametrized) zero sets and diffeomorphic bifurcation sets. Therefore, theorem 2 follows from lemma 16.

Proof. Polynomial (79) is a universal unfolding of the germ $|\zeta|^2\zeta + \varepsilon \bar{\zeta}^{q-1}$ under \mathbb{Z}_q contact equivalence. See [5] for a detailed computation. The tongue boundaries of a $p : q$ resonance are given by the bifurcation equations

$$\begin{aligned} G(\zeta, \mu) &= 0, \\ \det(dG)(\zeta, \mu) &= 0. \end{aligned}$$

As in [5, theorem 3.1] we put $u = |z|^2$ and $b(u, \mu) = a + i\delta + u$. Then $G(\zeta, \mu) = b(u, \mu)\zeta + \varepsilon \bar{\zeta}^{q-1}$. According to (the proof of) [5, theorem 3.1], the system of bifurcation

equations is equivalent to

$$\begin{aligned} |b|^2 &= \varepsilon^2 u^{q-2}, \\ \overline{b}b' + b\overline{b}' &= (q-2)\varepsilon^2 u^{q-3}, \end{aligned}$$

where $b' = \frac{\partial b}{\partial u}(u, \mu)$. A short computation reduces the latter system to the equivalent

$$\begin{aligned} (a+u)^2 + \delta^2 &= \varepsilon^2 u^{q-2}, \\ a+u &= \frac{1}{2}(q-2)\varepsilon^2 u^{q-3}. \end{aligned}$$

Eliminating u from this system of equations yields expression (80) for the tongue boundaries. \square

The discriminant set of the equivariant polynomial (79) forms the boundary of the resonance tongues. Also see figure 3. This concludes the proof of theorem 2.

Appendix D. The non-symmetric term of order q : proof of theorem 3

We now prove theorem 3 by deriving a precise expression for the coefficient $\gamma(v_0, 0)$ of $\varepsilon \bar{z}^{q-1} e^{ipt}$ in the normal form (18) for $\varepsilon = 0$. According to lemma 14, this coefficient is of the form

$$\gamma(\mu_0, 0) = d + \Psi_h,$$

where d is the coefficient of the term $\bar{z}^{q-1} e^{ipt}$ in the Taylor–Fourier expansion of $H(z, t, 0, \mu_0)$ at $z = 0$ and Ψ_h is a polynomial in the coefficients of the $(q-1)$ -jet of $h(\cdot, \mu_0)$ in z, \bar{z} at $z = 0$.

Claim 1. d is equal to the right-hand side of (23).

To prove this claim, we apply lemma 10 to conclude that the coefficient of d of $\bar{z}^{q-1} e^{ipt}$ in the Taylor–Fourier expansion of $H(z, t, 0, \mu_0)$ is of the form

$$d = c_1(f) C_1(P) + c_2(f) C_2(P) + C(P),$$

with

- $c_1(f)$ is the coefficient of \bar{z}^{q-1} in the Taylor expansion of $h_z(z, \mu_0)$ and $c_2(f)$ is the coefficient of \bar{z}^{q-1} in the Taylor expansion of $h_{\bar{z}}(z, \mu_0)$. In view of (51) and (52) these coefficients depend linearly on f and can be computed explicitly. Moreover, if f is a polynomial of degree less than q , we have $c_1(f) = c_2(f) = 0$.
- $C_1(P)$ is the coefficient of e^{ipt} in the Fourier expansion of $-\varphi_0(t, \mu_0)$ and $C_2(P)$ is the coefficient of e^{ipt} in the Fourier expansion of $-\varphi_0(t, \mu_0)$. In view of (45) and (50), these coefficients depend linearly on P , and can be computed explicitly.
- $C(P)$ is the coefficient of $\bar{z}^{q-1} e^{ipt}$ in the Taylor–Fourier expansion of $K(z, t, \mu_0)$. According to (9), this coefficient is given by (24) and depends linearly on P .

Moreover, the functional C is non zero and independent of C_1 and C_2 . Therefore, for fixed f the coefficient d is non zero for generic P .

Claim 2. Ψ_h only depends on the $(q - 1)$ -jet of $y \mapsto f(y, \mu_0)$ at $y = y_0$.

To prove this claim, recall that, according to lemma 9, h depends linearly on g . Therefore, the $(q - 1)$ -jet of $h(\cdot, \mu_0)$ in z, \bar{z} at $z = 0$ depends only on the $(q - 1)$ -jet of g at $z = 0$. Since g depends linearly on f , cf (14) and (41), it follows that the claim holds.

Taking $c_0(f) = \Psi_h$ concludes the proof of theorem 3. \square

References

- [1] Arnold V I 1982 *Geometrical Methods in the Theory of Ordinary Differential Equations* (Berlin: Springer)
- [2] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics* 2nd edn (New York: Springer)
- [3] Braaksma B L J, Broer H W and Huitema G B 1990 Toward a quasi-periodic bifurcation theory *Mem. AMS* vol 83 (Providence, RI: American Mathematical Society) pp 83–175
- [4] Broer H W 1980 Formal normal form theorems for vector fields and some consequences for bifurcations in the volume preserving case *Dynamical Systems and Turbulence (LNM)* vol 898 (Berlin: Springer) pp 54–74
- [5] Broer H W, Golubitsky M and Vegter G 2003 The geometry of resonance tongue: a singularity theory approach *Nonlinearity* **16** 1511–38
- [6] Broer H W, Golubitsky M and Vegter G 2007 Geometry of resonance tongues *Singularity Theory: Proc. 2005 Marseille Singularity School and Conf. (Marseille)* ed D Chéniot *et al* pp 327–56
- [7] Broer H W, Hanßmann H, Jorba A, Villanueva J and Wagener F O O 2003 Normal-internal resonances in quasi-periodically forces oscillators: a conservative approach *Nonlinearity* **16** 1751–91
- [8] Broer H W, Holtman S J and Vegter G 2008 Recognition of the bifurcation type of resonance in mildly degenerate Hopf–Neĭmark–Sacker families *Technical Report* Institute of Mathematics and Computer Science, University of Groningen, URL: <http://www.math.rug.nl/~gert/documents/2008/resonanceRecognitionBHV.pdf>
- [9] Broer H W, Hoveijn I, van Noort M, Simó C and Vegter G 2004 The parametrically forced pendulum: a case study in one-and-a-half degree of freedom *J. Dyn. Diff. Eqns* **16** 897–947
- [10] Broer H W and Levi M 1995 Geometrical aspects of stability theory for Hill's equations *Arch. Ration. Mech. Anal.* **13** 225–40
- [11] Broer H W, Puig J and Simó C 2003 Resonance tongues and instability pockets in the quasi-periodic Hill–Schrödinger equation *Commun. Math. Phys.* **241** 467–503
- [12] Broer H W and Sevryuk M B 2007 KAM theory: quasi-periodicity in dynamical systems *Handbook of Dynamical Systems* vol 3 ed H W Broer *et al* (Amsterdam: North-Holland) at press
- [13] Broer H W and Simó C 2000 Resonance tongues in hill's equations: a geometric approach *J. Diff. Eqns* **166** 290–327
- [14] Broer H W, Simó C and Tatjer J-C 1998 Towards global models near homoclinic tangencies of dissipative diffeomorphisms *Nonlinearity* **11** 667–770
- [15] Broer H W and Vegter G 1992 Bifurcational aspects of parametric resonance *Dynamics Reported (New Series)* vol 1 (Berlin: Springer) pp 1–51
- [16] Ciocci M C, Litvak-Hinenzon A and Broer H W 2005 Survey on dissipative Kam theory including quasi-periodic bifurcation theory based on lectures by Henk Broer *Geometric Mechanics and Symmetry: the Peyresq Lectures (LMS Lecture Notes Series)* vol 306 ed J Montaldi and T Ratiu (Cambridge: Cambridge University Press) pp 303–55
- [17] Mc Cullen N J, Mullin T and Golubitsky M 2007 Sensitive signal detection using a feed-forward oscillator network *Phys. Rev. Lett.* **98** 254101
- [18] Devaney R L 1989 *An Introduction to Chaotic Dynamical Systems* (Redwood City, CA: Addison-Wesley)
- [19] Elmhirst T and Golubitsky M 2006 Nilpotent Hopf bifurcations in coupled cell networks *SIAM J. Appl. Dyn. Syst.* **5** 205–51
- [20] Gambaudo J M 1985 Perturbation of a Hopf bifurcation by an external time-periodic forcing *J. Diff. Eqns* **57** 172–99
- [21] Golubitsky M, Nicol M and Stewart I 2004 Some curious phenomena in coupled cell networks *J. Nonlin. Sci.* **14** 207–36
- [22] Golubitsky M and Stewart I 2005 Synchrony versus symmetry in coupled cells *Equadiff 2003: Proc. Int. Conf. Differential Equations* (Singapore: World Scientific) pp 13–24
- [23] Golubitsky M, Stewart I and Török A 2005 Patterns of synchrony in coupled cell networks with multiple arrows *SIAM J. Appl. Dyn. Syst.* **4** 78–100
- [24] Guckenheimer J and Holmes Ph 1983 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Applied Mathematical Sciences vol 42)* (New York: Springer)

- [25] Hopf E 1948 A mathematical example displaying features of turbulence *Commun. Appl. Math.* **1** 303–22
- [26] Hopf E 1956 Repeated branching through loss of stability, an example *Proc. Conf. on Differential Equations (Dedicated to A Weinstein)* (College Park, MD: University Maryland Book Store) ed J B Diaz pp 49–56
- [27] Kuznetsov Y A 1998 *Elements of Applied Bifurcation Theory (Applied Mathematical Sciences vol 112, 2nd edn)* (New York: Springer)
- [28] Landau L D 1944 On the problem of turbulence *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **44** 311–14
- [29] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* 2nd edn (Oxford: Pergamon)
- [30] Ruelle D and Takens F 1971 Note concerning our paper On the nature of turbulence *Commun. Math. Phys.* **23** 343–44
- [31] Ruelle D and Takens F 1971 On the nature of turbulence *Commun. Math. Phys.* **20** 167–92
- [32] Takens F 1974 Singularities of vector fields *Publ. Math. IHES* **43** 48–100
- [33] Takens F 2001 Forced oscillations and bifurcations *Global Analysis of Dynamical Systems, Festschrift dedicated to Floris Takens on his 60th birthday* ed B Krauskopf *et al* (Bristol: Institute of Physics Publishing) pp 1–61